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A class of slippage tests

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1. Summary.

In this report some slippage tests for variates following various specified distributions, viz. the normal, the Poisson, the binomial and the negative binomial distribution, as well as a slippage test for the method of m rankings and a distribution free k-sample slippage test, are discussed. A method for obtaining approximate critical values at a prescribed significance level ε, such that the true significance level corresponding to these values lies between ε and $\varepsilon - \frac{1}{2}\varepsilon^2$, is found to be applicable in all cases under consideration. The same approximation was applied before by W.G. COCHRAN (1941), R. DOORNBOS (1956) and R. DOORNBOS and H.J. PRINS (1956) to slippage tests for gamma-variates. In addition decision procedures are given to select the slipped variate when we reject that none of the variates has slipped.

In some cases power functions of the tests and optimum properties of the decision procedures are also considered.

2. Introduction; description of the tests.

All the tests dealt with in this report are of the following type. Suppose we have k random variables [1])

(2.1) $$\underline{x}_1, \ldots, \underline{x}_k ,$$

which are, under H_0, the hypothesis tested, distributed simultaneously with some distribution function $F(x_1, \ldots, x_k)$, which may be continuous or not.

Suppose the observed values of $\underline{x}_1, \ldots, \underline{x}_k$ are respectively $x_1, \ldots, x_k$. When testing against slippage to the right we determine the right hand tail probabilities

(2.2) $$d_j \overset{\text{def}}{=} P\left[\underline{x}_j \geq x_j\right] , \quad (j=1, \ldots, k) \; .^{2)}$$

We reject H_0 and decide that the m-th population has slipped to the right if

(2.3) $$d_m = \min_j d_j \leq \varepsilon/k.$$

Testing against slippage to the right requires computing

(2.4) $$e_j = P\left[\underline{x}_j \leq x_j\right] , \quad (j=1, \ldots, k).$$

Now H_0 is rejected and it is concluded that the m-th population has slipped to the left if

(2.5) $$e_m = \min_j e_j \leq \varepsilon/k.$$

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1) Random variables are denoted by underlined symbols.

2) The symbol $\overset{\text{def}}{=}$ denotes an equality, defining the left hand member.

Consider now a set of k real numbers $g_1,\ldots,g_k$ and the probabilities defined by

$$(2.6)\qquad \begin{cases} p_i \overset{\text{def}}{=} P\left[\underline{x}_i \leqq g_i\right], \\ p_{i,j} \overset{\text{def}}{=} P\left[\underline{x}_i \leqq g_i \text{ and } \underline{x}_j \leqq g_j\right], \quad (i \neq j), \\ q_i \overset{\text{def}}{=} P\left[\underline{x}_i > g_i\right], \\ q_{ij} \overset{\text{def}}{=} P\left[\underline{x}_i > g_i \text{ and } \underline{x}_j > g_j\right], \ (i \neq j), \end{cases}$$

all computed under H_o.

Denoting by P the probability that at least one of the $\underline{x}_i$ does not exceed the corresponding value g_i, it follows from BONFERRONI's inequality (cf. W. FELLER (1950), chapter 4) that

$$(2.7)\qquad \sum_i p_i - \sum_{i<j} p_{i,j} \leqq P \leqq \sum_i p_i .$$

For Q, the probability that at least one $\underline{x}_i$ exceeds g_i, we have

$$(2.8)\qquad \sum_i q_i - \sum_{i<j} q_{i,j} \leqq Q \leqq \sum_i q_i .$$

Then in each case separately we proceed to prove the inequality

$$(2.9)\qquad p_{i,j} \leqq p_i p_j ,$$

or

$$(2.10)\qquad q_{i,j} \leqq q_i q_j ,$$

which is equivalent with (2.9) (cf. R. DOORNBOS and H.J. PRINS (1956)). Of course (2.9) and (2.10) do only hold for a class of distribution functions $F(x_1,\ldots,x_k)$. The problem of finding general conditions imposed on $F(x_1,\ldots,x_k)$, sufficient for the validity of (2.9) has only partly been solved in this report. Besides in some cases (2.9) only holds for some sets $g_1,\ldots,g_k$ for instance for all $g_i \geqq 0$.

Assuming that (2.9) and (2.10) are true we get immediately from (2.7) and (2.8) respectively

$$(2.11)\qquad \sum_i p_i - \sum_{i<j} p_i p_j \leqq P \leqq \sum_i p_i$$

and

$$(2.12)\qquad \sum_i q_i - \sum_{i<j} q_i q_j \leqq Q \leqq \sum_i q_i$$

respectively. Denoting $\sum_i p_i$ by p (p needs not be $\leqq 1$) we have

$$p^2 = \left(\sum_i p_i\right)^2 = 2\sum_{i<j} p_i p_j + \sum_i p_i^{\,2} \geqq 2\sum_{i<j} p_i p_j ,$$

where the equality sign only holds if all p_i vanish, or

$$\sum_{i<j} p_i p_j \leqq \tfrac{1}{2} p^2 .$$

Thus

$$(2.13) \qquad p - \tfrac{1}{2}p^2 \leqq P \leqq p$$

and

$$(2.14) \qquad q - \tfrac{1}{2}q^2 \leqq Q \leqq q,$$

when $\sum_i q_i = q$.

Now, when testing H_o against "slippage to the left" of one of the k variables the critical region is of the form $\{x_1 \leqq g_{1\varepsilon}, \ldots,$ or $x_k \leqq g_{k\varepsilon}\}$.
The values $g_{i\varepsilon}$ are determined so as to make all p_i equal to ε/k, where ε is the prescribed level of significance. In the discontinuous case this will in general not be possible; there $g_{i\varepsilon}$ is the largest value which can be attained by $\underline{x}_i$ with a positive probability, satisfying

$$(2.15) \qquad \varepsilon_i' = P\left[\underline{x}_i \leqq g_{i,\varepsilon}\right] \leqq \varepsilon/k.$$

So from (2.13) it follows that the probability P_ε of rejecting H_o unjustly satisfies

$$(2.16) \qquad \varepsilon - \tfrac{1}{2}\varepsilon^2 \leqq P_\varepsilon \leqq \varepsilon,$$

or

$$(2.17) \qquad \varepsilon' - \tfrac{1}{2}(\varepsilon')^2 \leqq P_\varepsilon \leqq \varepsilon' \quad (\varepsilon' = \sum_i \varepsilon_i')$$

respectively, accordingly as the continuous or the discontinuous case is considered.

Testing "slippage to the right" we get similar bounds for Q_ε, the probability of rejecting H_o when H_o is true.

3. The slippage test for normal distributions.

We consider k normal distributions with unknown means $\mu_1, \mu_2, \ldots, \mu_k$ and common unknown variance σ^2. From these distributions we have samples of $n_1, n_2, \ldots, n_k$ independent observations respectively.

We want to test the hypothesis

$$(3.1) \qquad H_o: \mu_1 = \ldots = \mu_k = \mu, \text{ say},$$

against the alternatives

$$(3.2) \qquad \begin{cases} H_1: \mu_1 = \ldots = \mu_{i-1} = \mu_{i+1} = \ldots = \mu_k = \mu \\ \mu_i = \mu + \Delta \quad (\Delta > 0), \end{cases}$$

for one value of i, which is, however, not known, or

$$(3.3) \qquad \begin{cases} H_2: \mu_1 = \ldots = \mu_{i-1} = \mu_{i+1} = \ldots = \mu_k = \mu \\ \mu_i = \mu - \Delta \, (\Delta > 0), \end{cases}$$

for one unknown value of i. From the observations

$$(3.4)\qquad \begin{cases} \underline{y}_{11},\ldots,\underline{y}_{1n_1}, \\ \underline{y}_{21},\ldots,\underline{y}_{2n_2}, \\ \\ \underline{y}_{k1},\ldots,\underline{y}_{kn_k}, \end{cases}$$

the variables

$$(3.5)\qquad \underline{b}_i = \frac{\sqrt{n_i}\,(\underline{y}_i-\underline{y})}{\sqrt{\sum\limits_j n_j(\underline{y}_j-\underline{y})^2+\sum\limits_{j,l}(\underline{y}_{jl}-\underline{y}_j)^2}}, \quad (i=1,\ldots,k).$$

are formed, where

$$(3.6)\qquad \begin{cases} \underline{y}_i = \dfrac{1}{n_i}\sum\limits_l \underline{y}_{il}, \\ \underline{y} = \dfrac{1}{\sum\limits_j n_j}\sum\limits_{j,l}\underline{y}_{jl} = \dfrac{1}{\sum\limits_j n_j}\sum\limits_j n_j y_j . \end{cases}$$

The $\underline{b}_i$ take the place of the variables $\underline{x}_i$ in (2.1).
In the following section we shall prove the inequality corresponding to (2.9) if g_i and g_j have the same sign and it will be proved that

$$(3.7)\qquad \underline{u}_i = \tfrac{1}{2}\left(1+\sqrt{\frac{\sum n_j}{\sum n_j-n_i}}\,\underline{b}_i\right)$$

has a B-distribution with parameters $\frac{N+k-2}{2}$ and $\frac{N+k-2}{2}$, where N is defined by

$$(3.8)\qquad N = \sum n_j - k;$$

or, equivalently, that

$$(3.9)\qquad \underline{t}_i = \sqrt{N+k-2}\,\frac{\sqrt{\dfrac{\sum n_j}{\sum n_j-n_i}}\,\underline{b}_i}{\sqrt{\left(1-\dfrac{\sum n_j}{\sum n_j-n_i}\,\underline{b}_i^2\right)}},$$

has a Student's t-distribution with N+k-2 degrees of freedom, for $i=1,\ldots,k$.

Thus the procedure described in section 2 can be applied and the d_j and e_j values as defined by (2.2) and (2.4) may be obtained for instance by means of (3.7) and the methods described in section 6 of R. DOORNBOS and H.J. PRINS (1956).

In the present case the determination of the minimum d and e values is much simpler however because these minimum values correspond to respectively the largest and the smallest of the u_i and thus of the $\sqrt{\frac{\sum n_j}{\sum n_j-n_i}}\,b_i$ and consequently only one incomplete B-integral has to be computed. The critical values $g_{i\varepsilon}$ for the $\underline{b}_i$ are determined from

$$(3.10) \qquad g_{i,\varepsilon} = \sqrt{\frac{\sum n_j - n_i}{\sum n_j}}\,(2u_{\varepsilon/k} - 1),$$

where $u_{\varepsilon/k}$ is defined by

$$(3.11) \qquad P\left[\underline{u}_i \leqq u_{\varepsilon/k}\right] = \varepsilon/k.$$

Because of the symmetry of the distribution of $\underline{u}_i$ with respect to the point $\frac{1}{2}$, the critical values $G_{i,\varepsilon}$ for the test against slippage to the right are

$$(3.12) \qquad G_{i,\varepsilon} = \sqrt{\frac{\sum n_j - n_i}{\sum n_j}}\,(2u_{1-\varepsilon/k} - 1) = -g_{i,\varepsilon}.$$

In the most simple case, i.e. $n_1 = \ldots = n_k = 1$, our test-statistic reduces to the one suggested already by E.S. PEARSON and C. CHANDRA SEKAR (1936) but for a constant factor. Using previous work of W.R. THOMPSON (1935), who derived in this special case the distribution of $\underline{t}_i$ as defined by (3.9), PEARSON and CHANDRA SEKAR were able to derive certain percentage points of max $\underline{b}_i$ and min $\underline{b}_i$ without deriving the exact distribution. They used the same approximation as is done here, but only up to $g_{1\varepsilon} = \ldots = g_{k\varepsilon} = g_\varepsilon \leqq -\sqrt{\frac{k-2}{2k}}$ (or $G_\varepsilon \geqq \sqrt{\frac{k-2}{2k}}$), because, if all n_i are equal, in that region the probability that two of the variables, e.g. $\underline{b}_i$ and $\underline{b}_j$, both do not exceed g_ε or exceed G_ε is equal to zero. Thus the level of significance is then exactly equal to ε.

The exact distribution for $n_1 = \ldots = n_k = 1$ has been computed numerically by F.E. GRUBBS (1950), who gave tables of exact percentage points up to k=25 for ε =0.10, 0.05, 0.025 and 0.01.

E. PAULSON (1952) proposed the same test statistic (but for a constant factor) for slippage to the right and the same approximation as suggested here in the special case $n_1 = \ldots = n_k = n$, but he gives no bounds for the corresponding level of significance. PAULSON proved that in this case the use of max $\underline{b}_i$ as test-statistic has the following optimum property. Let D_0 denote the decision that the k means are equal and let D_j (j=1,...,k) denote the decision that D_0 is incorrect and that $\mu_j = \max(\mu_1, \ldots, \mu_k)$. Now the procedure:

$$(3.13) \qquad \begin{cases} \text{if } \underline{b}_m > \lambda_\varepsilon, \text{ select } D_m, \\ \text{if } \underline{b}_m \leqq \lambda_\varepsilon, \text{ select } D_0, \end{cases}$$

where m is the index of the maximum b-value maximizes the probability of making a correct decision, subject to the following restrictions.

(a) when all means are equal, D_0 should be selected with probability $1-\varepsilon$,

(b) the decision procedure must be invariant if a constant is added to the observations,

(c) the decision procedure must be invariant when all the observations are multiplied by a positive constant, and

(d) the decision procedure must be symmetric in the sense that the probability of making a correct decision when the i-th mean has slipped to the right by an amount Δ must be the same for $i=1,2,\ldots,k$.

The constant λ_ε in (3.13) is determined by requirement (a). Our critical value G_ε is an approximation of λ_ε.

The case of slippage to the left, although not mentioned explicitely by PAULSON is completely analogous and the same optimum property holds there.

4. Proof of the results stated in 3.

In this section we shall prove the inequality

$$(4.1)\qquad P\left[\underline{b}_i \leqq g_i \text{ and } \underline{b}_j \leqq g_j\right] \leqq P\left[\underline{b}_i \leqq g_i\right] \cdot P\left[\underline{b}_j \leqq g_j\right], \text{ provided } g_i g_j \geqq 0,$$

where $\underline{b}_i$ and $\underline{b}_j$ are defined by (3.5), for all pairs i,j ($i \neq j$; $i,j=1,\ldots,k$). Obviously there is no loss of generality in taking $i=1$ and $j=2$.

First we shall derive the simultaneous distribution of $\underline{b}_1$ and $\underline{b}_2$. We transform the variables $\underline{y}_1,\ldots,\underline{y}_k$, as defined by (3.6) into $\underline{a}_1,\ldots,\underline{a}_{k-2}$, $\underline{y}$, $\underline{s}_1$, where

$$(4.2)\qquad \begin{cases} \underline{a}_j = \dfrac{\sqrt{n_j}(\underline{y}_j-\underline{y})}{\sqrt{\sum n_i(\underline{y}_i-\underline{y})^2}}, & (j=1,\ldots,k), \\ \underline{s}_1^2 = \sum n_i(\underline{y}_i-\underline{y})^2. \end{cases}$$

There is no one-to-one correspondence between the points $(y_1,\ldots,y_k)$ and $(a_1,\ldots,a_{k-2},s_1,y)$, for, if $\sqrt{n_{k-1}}(y_{k-1}-y)$ is replaced by $\sqrt{n_k}(y_k-y)$ and reversely, we obtain the same set of values $(a_1,\ldots,a_{k-2},s_1,y)$. Therefore we divide the y-space into two parts R_1 and R_2 such that in R_1 $\sqrt{n_{k-1}}(y_{k-1}-y) > \sqrt{n_k}(y_k-y)$ and in R_2 $\sqrt{n_{k-1}}(y_{k-1}-y) \leqq \sqrt{n_k}(y_k-y)$, then in both parts the correspondence is unique in both senses (cf. H. CRAMER (1946), section 22.2). In both sub-spaces we shall compute the Jacobian denoted respectively by J_1 and J_2. From (4.2) follows that

$$(4.3)\qquad \begin{cases} \sum_1^k \sqrt{n_j}\, a_j = 0, \\ \sum_1^k a_j^2 = 1, \end{cases}$$

so after some calculation it is found that

$$(4.4)\qquad a_{k-1}=\frac{-\sqrt{n_{k-1}}\sum_{1}^{k-2}\sqrt{n_i}a_i\pm\sqrt{n_k}\sqrt{(1-\sum_{1}^{k-2}a_i^{2})(n_{k-1}+n_k)-(\sum_{1}^{k-2}\sqrt{n_i}a_i)^{2}}}{n_{k-1}+n_k}$$

and

$$(4.5)\qquad a_k=\frac{-\sqrt{n_k}\sum_{1}^{k-2}\sqrt{n_i}a_i\mp\sqrt{n_{k-1}}\sqrt{(1-\sum_{1}^{k-2}a_i^{2})(n_{k-1}+n_k)-(\sum_{1}^{k-2}\sqrt{n_i}a_i)^{2}}}{n_{k-1}+n_k}$$

The signs occurring in the expressions(4.4) and (4.5) are determined by the requirement that in R_1 $a_k > a_{k-1}$, whereas in R_2 $a_k \leqq a_{k-1}$. The Jacobian J becomes

$$(4.6)\qquad J=\begin{vmatrix}
\frac{s_1}{\sqrt{n_1}} & 0 & \dots & 0 & \frac{a_1}{\sqrt{n_1}} & 1\\
0 & \frac{s_1}{\sqrt{n_2}} & \dots & 0 & \frac{a_2}{\sqrt{n_2}} & 1\\
\vdots & & & & & \\
0 & 0 & \dots & \frac{s_1}{\sqrt{n_{k-2}}} & \frac{a_{k-2}}{\sqrt{n_{k-2}}} & 1\\
\frac{s_1}{\sqrt{n_{k-1}}}\cdot\frac{\partial a_{k-1}}{\partial a_1} & \frac{s_1}{\sqrt{n_{k-1}}}\cdot\frac{\partial a_{k-1}}{\partial a_2} & \dots & \frac{s_1}{\sqrt{n_{k-1}}}\cdot\frac{\partial a_{k-1}}{\partial a_{k-2}} & \frac{a_{k-1}}{\sqrt{n_{k-1}}} & 1\\
\frac{s_1}{\sqrt{n_k}}\cdot\frac{\partial a_k}{\partial a_1} & \frac{s_1}{\sqrt{n_k}}\cdot\frac{\partial a_k}{\partial a_2} & \dots & \frac{s_1}{\sqrt{n_k}}\cdot\frac{\partial a_k}{\partial a_{k-2}} & \frac{a_k}{\sqrt{n_k}} & 1
\end{vmatrix}.$$

Now $\frac{\partial a_{k-1}}{\partial a_i}$ can be derived from (4.4) and further it is easily seen that $\frac{\partial a_k}{\partial a_i} = -\frac{1}{\sqrt{n_k}}\left(\sqrt{n_i}+\sqrt{n_{k-1}}\frac{\partial a_{k-1}}{\partial a_i}\right)$. Substituting these expressions into (4.6) it is found after some calculation that

$$(4.7)\qquad J=\frac{\pm\sum_{1}^{k}n_i\;s_1^{\,k-2}}{\sqrt{\prod_{1}^{k}n_i}\sqrt{(1-\sum_{1}^{k-2}a_i^{2})(n_{k-1}+n_k)-(\sum_{1}^{k-2}\sqrt{n_i}a_i)^{2}}},$$

both in R_1 and R_2.

The joint distribution of $\underline{y}_1,\dots,\underline{y}_k$, under H_0, is, both in R_1 and in R_2, given by their density function

$$(4.8)\qquad f_1(y_1,\dots,y_k)=\frac{\prod_{1}^{k}\sqrt{n_i}}{(2\pi\sigma^2)^{k/2}}\,e^{-\frac{1}{2\sigma^2}\sum_{1}^{k}n_i(y_i-\mu)^2}$$
$$=\frac{\prod_{1}^{k}\sqrt{n_i}}{(2\pi\sigma^2)^{k/2}}\,e^{-\frac{1}{2\sigma^2}\left\{\sum_{1}^{k}n_i(y_i-y)^2+\sum_{1}^{k}n_i(y-\mu)^2\right\}}.$$

Consequently the density function of $\underline{a}_1,\dots,\underline{a}_{k-2}$, $\underline{s}_1,\underline{y}$ is given by

$$(4.9) \quad f_2(a_1,\ldots,a_{k-2},s_1,y) = \left\{ |J_1| + |J_2| \right\} f_1 =$$
$$= \frac{2\sum n_i \, s_1^{k-2}}{(2\pi\sigma^2)^{k/2}} \, \frac{e^{-\frac{s_1^2}{2\sigma^2} - \frac{\sum n_i}{2\sigma^2}(y-\mu)^2}}{\sqrt{(1-\sum_1^{k-2} a_i^2)(n_{k-1}+n_k)-(\sum_1^{k-2}\sqrt{n_i}a_i)^2}},$$

if $(1-\sum_1^{k-2} a_i^2)(n_{k-1}+n_k)-(\sum_1^{k-2}\sqrt{n_i}\, a_i)^2 \geqq 0$ and zero otherwise. Where in the following it is obvious in what domain a density function is defined it will not always be stated explicitely.

Thus we see that $\underline{s}_1$ and $\underline{y}$ are mutually independent and independent of $\underline{a}_1,\ldots,\underline{a}_{k-2}$. The distribution functions of $\underline{s}_1$ and $\underline{y}$ are well known, so from (4.9) we get immediately the density function of $\underline{a}_1,\ldots,\underline{a}_{k-2}$.

$$(4.10) \qquad f_3(a_1,\ldots,a_{k-2}) = \frac{\sqrt{\sum n_i} \quad \Gamma(\frac{k-1}{2})}{\pi^{(k-1)/2}\sqrt{(n_{k-1}+n_k)(1-\sum_1^{k-2}a_i^2)-(\sum_1^{k-2}\sqrt{n_i}a_i)^2}}.$$

Next we introduce the variables

$$(4.11) \qquad \underline{a}'_2,\ldots,\underline{a}'_k,$$

defined by

$$(4.12) \qquad \underline{a}'_j = \frac{\sqrt{n_j}(\underline{y}_j-\underline{y}')}{\underline{s}'_1}, \quad (j=2,\ldots,k),$$

where

$$(4.13) \qquad \begin{cases} \underline{y}' = \dfrac{1}{\sum_2^k n_i} \sum_2^k n_i \, \underline{y}_i, \\ \underline{s}'_1 = \sqrt{\sum_2^k n_i(\underline{y}_i-\underline{y}')^2}. \end{cases}$$

Straightforward computation shows that $\underline{a}'_j$ can be written as follows

$$(4.14) \qquad \underline{a}'_j = \frac{\underline{a}_j + \frac{\sqrt{n_j}}{\sum n_i - n_1}\sqrt{n_1}\,\underline{a}_1}{\sqrt{1-\frac{\sum n_i}{\sum n_i - n_1}a_1^2}}, \quad (j=2,\ldots,k).$$

The density function of $\underline{a}_1,a'_2,\ldots,\underline{a}'_{k-2}$ is found to be

$$(4.15) \quad f_4(a_1,a'_2,\ldots,a'_{k-2}) =$$
$$= \frac{\sqrt{\sum_1^k n_i} \;\; \Gamma(\frac{k-1}{2})}{\pi^{(k-1)/2}} \; \frac{\left(1-\frac{\sum n_i}{\sum n_i - n_1}a_1^2\right)^{\frac{k-4}{2}}}{\sqrt{(n_{k-1}+n_k)(1-\sum_2^{k-2}(a'_i)^2)-(\sum_2^{k-2}a'_i\sqrt{n_i})^2}}.$$

So $\underline{a}_1$ is independent of $\underline{a}'_2,\ldots,\underline{a}'_{k-2}$ simultaneously and consequently also of $\underline{a}'_2$ alone. From (4.15) it is found that the density function of $\underline{a}_1$ reads

$$(4.16)\qquad f(a_1) = \sqrt{\frac{\sum n_i}{\sum n_i - n_1}}\;\frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k-2}{2})}\;\frac{1}{\sqrt{\pi}}\;\left(1-\frac{\sum n_i}{\sum n_i-n_1}\,a_1^2\right)^{\frac{k-4}{2}},$$

$$\left(-\sqrt{\frac{\sum n_i - n_1}{\sum n_i}} \leqq a_1 \leqq \sqrt{\frac{\sum n_i - n_1}{\sum n_i}}\right).$$

Because $\underline{a}'_2,\ldots\underline{a}'_{k-2}$ are the same functions of $\underline{y}_2,\ldots,\underline{y}_k$ as $\underline{a}_1,\ldots,\underline{a}_{k-2}$ are of $\underline{y}_1,\ldots,\underline{y}_k$, the density function of $\underline{a}'_2$ has the same form with k replaced by k-1, $\sum n_i$ bij $\sum n_i - n_1$ and $\sum n_i - n_1$ by $\sum n_i - n_1 - n_2$. Because $\underline{a}_1$ and $\underline{a}'_2$ are independent their joint distribution and consequently the joint distribution of $\underline{a}_1$ and $\underline{a}_2$ follows easily, using the transformation (4.14) with j=2. It is found to be

$$(4.17)\qquad g(a_1,a_2) = \sqrt{\frac{\sum n_i}{\sum n_i - n_1 - n_2}}\;\frac{k-3}{2\pi}\;.$$

$$.\left\{1-\frac{\sum n_i - n_2}{\sum n_i - n_1 - n_2}\,a_1^2 - \frac{2\sqrt{n_1 n_2}}{\sum n_i - n_1 - n_2}\,a_1 a_2 - \frac{\sum n_i - n_1}{\sum n_i - n_1 - n_2}\,a_2^2\right\}^{\frac{k-5}{2}}.$$

The function $g(a_1,a_2)$ is valid in the domain where the expression between braces is positive.

Returning now to the $\underline{b}_i$ it is seen from (3.5) that

$$(4.18)\qquad \underline{b}_i = \frac{\underline{a}_i}{\sqrt{1+\frac{\underline{s}^2}{\underline{s}_1^2}}}\;,$$

where

$$(4.19)\qquad \underline{s}^2 = \sum_{j,l}(\underline{y}_{jl} - \underline{y}_j)^2\;.$$

As is well known $\underline{s}^2$ is distributed independently of $\underline{y}_1,\ldots,\underline{y}_k$ and consequently of $\underline{a}_1,\ldots,\underline{a}_{k-2}$ and $\underline{s}_1^2$ simultaneously. Further $\underline{s}^2/\sigma^2$ has a χ^2 distribution with $N(=\sum n_i - k)$ degrees of freedom and $\underline{s}_1^2/\sigma^2$ a χ^2 distribution with k-1 degrees of freedom, while $\underline{s}_1^2$ is also independent of $\underline{a}_1,\ldots,\underline{a}_{k-2}$ (cf. (4.9)). So

$$(4.20)\qquad \underline{F} = \frac{k-1}{N}\;\frac{\underline{s}^2}{\underline{s}_1^2} = \frac{k-1}{N}\,.\underline{G},\ \text{say},$$

has FISHER's F-distribution with N and k-1 degrees of freedom, while $\underline{F}$ and consequently also $\underline{G}$ are independent of $\underline{a}_1,\ldots,\underline{a}_{k-2}$ simultaneously.

The density function of $\underline{G}$ is known to be

$$(4.21)\qquad f_5(G) = \frac{\Gamma(\frac{N+k-1}{2})}{\Gamma(\frac{N}{2})\,\Gamma(\frac{k-1}{2})}\;\frac{G^{\frac{N-2}{2}}}{(1+G)^{\frac{N+k-1}{2}}}\,.$$

So the joint density function of $\underline{a}_1$, $\underline{a}_2$ and $\underline{G}$ is

$$(4.22)\qquad f_6(a_1,a_2,G) = \sqrt{\frac{\sum n_i}{\sum n_i-n_1-n_2}}\;\frac{\Gamma(\frac{N+k-1}{2})}{\Gamma(\frac{N}{2})\,\Gamma(\frac{k-3}{2})}\;\frac{1}{\pi}\;\frac{G^{\frac{N-2}{2}}}{(1+G)^{\frac{N+k-1}{2}}}$$

$$\left\{1-\frac{\sum n_i-n_2}{\sum n_i-n_1-n_2}\,a_1^2-\frac{2\sqrt{n_1n_2}}{\sum n_i-n_1-n_2}\,a_1a_2-\frac{\sum n_i-n_1}{\sum n_i-n_1-n_2}\,a_2^2\right\}^{\frac{k-5}{2}}.$$

We have

$$(4.23)\qquad \begin{cases} a_1 = \sqrt{1+G}\; b_1, \\ a_2 = \sqrt{1+G}\; b_2. \end{cases}$$

The joint distribution of $\underline{b}_1$, $\underline{b}_2$ and $\underline{G}$ becomes

$$(4.24)\qquad f_7(b_1,b_2,G) = \sqrt{\frac{\sum n_i}{\sum n_i-n_1-n_2}}\;\frac{\Gamma(\frac{N+k-1}{2})}{\Gamma(\frac{N}{2})\,\Gamma(\frac{k-3}{2})}\;\frac{1}{\pi}\;\frac{G^{\frac{N-2}{2}}}{(1+G)^{\frac{N+k-3}{2}}}$$

$$\left\{1-(1+G)\left[\frac{\sum n_i-n_2}{\sum n_i-n_1-n_2}\,b_1^2+\frac{2\sqrt{n_1n_2}}{\sum n_i-n_1-n_2}\,b_1b_2+\frac{\sum n_i-n_1}{\sum n_i-n_1-n_2}\,b_2^2\right]\right\}^{\frac{k-5}{2}}.$$

The joint density function of $\underline{b}_1$ and $\underline{b}_2$ is equal to

$$(4.25)\qquad h(b_1,b_2) = \int_0^\infty f_7(b_1,b_2,G)\,dG.$$

This integral has the form

$$(4.26)\qquad I = c_1\int_0^{\frac{1}{c}-1} \frac{\{1-c(1+G)\}^a . G^b}{(1+G)^{a+b+2}}\,dG.$$

In (4.26) we make the substitution

$$(4.27)\qquad 1+G = \frac{1}{(1-c)v+c}\,,$$

which gives for (4.26)

$$(4.28)\qquad I = c_1(1-c)^{a+b+1}\int_0^1 v^a(1-v)^b dv =$$

$$= c_1(1-c)^{a+b+1}\;\frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}\,.$$

Applying this to (4.25), where

$$(4.29)\quad \begin{cases} c = \dfrac{\sum n_i - n_2}{\sum n_i - n_1 - n_2} b_1^2 + \dfrac{2\sqrt{n_1 n_2}}{\sum n_i - n_1 - n_2} b_1 b_2 + \dfrac{\sum n_i - n_1}{\sum n_i - n_1 - n_2} b_2^2 , \\ a = \dfrac{k-5}{2} , \\ b = \dfrac{N-2}{2} , \\ c_1 = \sqrt{\dfrac{\sum n_i}{\sum n_i - n_1 - n_2}} \; \dfrac{\Gamma(\frac{N+k-1}{2})}{\pi \Gamma(\frac{N}{2}) \Gamma(\frac{k-3}{2})} , \end{cases}$$

we find

$$(4.30)\quad h(b_1, b_2) = \sqrt{\frac{\sum n_i}{\sum n_i - n_1 - n_2}} \; \frac{N+k-3}{2\pi} \left\{ 1 - \frac{\sum n_i - n_2}{\sum n_i - n_1 - n_2} b_1^2 + \right.$$
$$\left. - \frac{2\sqrt{n_1 n_2}}{\sum n_i - n_1 - n_2} b_1 b_2 - \frac{\sum n_i - n_1}{\sum n_i - n_1 - n_2} b_2^2 \right\}^{\frac{N+k-5}{2}} ,$$

if the expression between braces is positive and $h(b_1, b_2)$ is zero otherwise.

If we apply the transformation

$$(4.31)\qquad \underline{b}_2' = \frac{\underline{b}_2 + \dfrac{\sqrt{n_2}}{\sum n_i - n_1} \sqrt{n_1} \underline{b}_1}{\sqrt{1 - \dfrac{\sum n_i}{\sum n_i - n_1} \underline{b}_1^2}} ,$$

analogous to (4.14), it appears that $\underline{b}_2'$ and $\underline{b}_1$ are independently distributed and that the density function of $\underline{b}_1$ is given by

$$(4.32)\quad p(b_1) = \sqrt{\frac{\sum n_i}{\sum n_i - n_1}} \; \frac{\Gamma(\frac{N+k-1}{2})}{\Gamma(\frac{N+k-2}{2})} \; \frac{1}{\sqrt{\pi}} \left\{ 1 - \frac{\sum n_i}{\sum n_i - n_1} b_1^2 \right\}^{\frac{N+k-4}{2}}$$

and that $\underline{b}_2'$ has a distribution of the same form with k replaced by k-1, $\sum n_i$ by $\sum n_i - n_1$ and $\sum n_i - n_1$ by $\sum n_i - n_1 - n_2$. It is easily seen that (4.32) can be transformed into a symmetric B-distribution or into a t-distribution by applying respectively the transformations (3.7) or (3.9) for i = 1.

The region where $h(b_1, b_2)$ differs from zero is bounded by an ellipse (cf. fig. 4.1) with principle axes of length 1 and $\sqrt{\dfrac{\sum n_i - n_1 - n_2}{\sum n_i}}$, whose directions are given respectively by the lines

$$(4.33)\qquad \begin{cases} n_1 b_1 + \sqrt{n_1 n_2} b_2 = 0, \\ -\sqrt{n_1 n_2} b_1 + n_1 b_2 = 0. \end{cases}$$

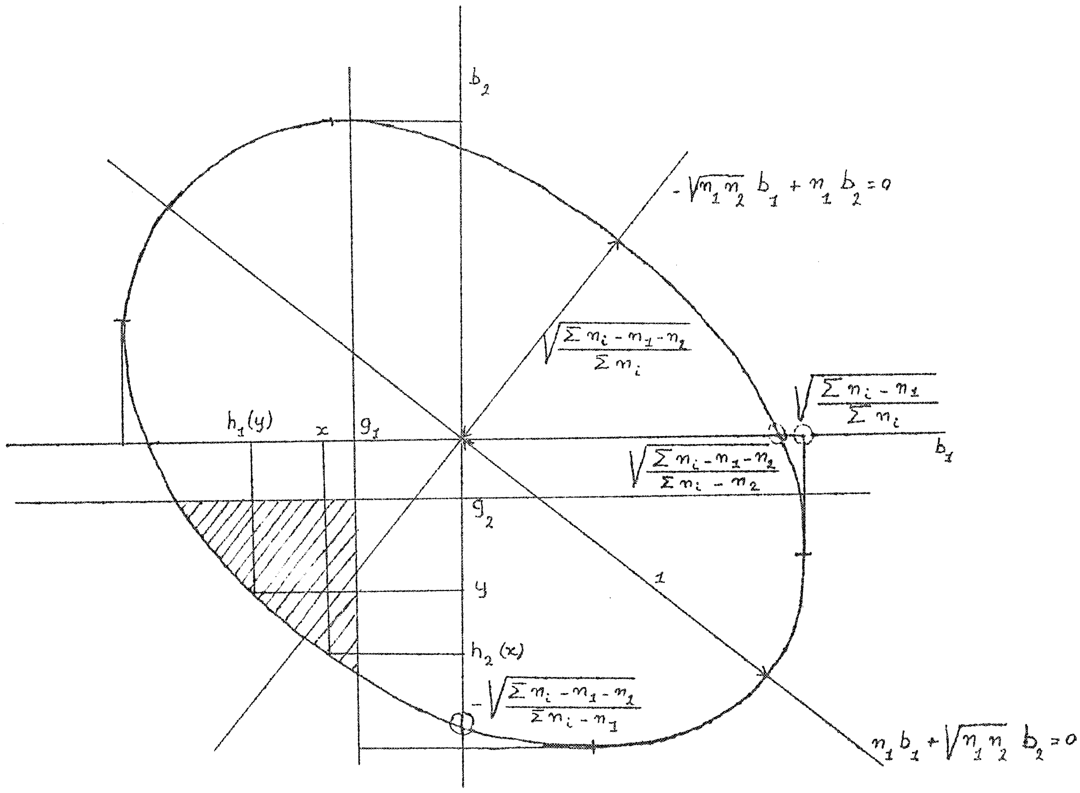

Figure 4.1
The region where $h(b_1,b_2) > 0$

We now proceed to prove the inequality (4.1). We suppose that both g_1 and g_2 are $\leqq 0$. This is no restriction for when (4.1) holds for a pair of values g_1 and g_2, the inequality $P[\underline{b}_1 > -g_1 \text{ and } \underline{b}_2 > -g_2] \leqq$ $\leqq P[\underline{b}_1 > -g_1] . P[\underline{b}_2 > -g_2]$ holds also for reasons of symmetry. Consequently (4.1) is also true for $-g_1$ and $-g_2$ because of the equivalence of (2.9) and (2.10). Further we may assume that the point (g_1, g_2) lies within the ellipse of figure 4.1, because otherwise $P[\underline{b}_1 \leqq g_1 \text{ and } \underline{b}_2 \leqq g_2] = 0$ and (4.1) is obviously fulfilled. We shall prove that in the (g_1,g_2)-region considered (4.1) holds with the $<$ sign.

We put

$$(4.34) \quad \begin{cases} c_1 \overset{\text{def}}{=} \sqrt{\dfrac{\sum n_i}{\sum n_i - n_1}} \; \dfrac{\Gamma(\frac{N+k-1}{2})}{\Gamma(\frac{N+k-2}{2})} \; \dfrac{1}{\sqrt{\pi}} , \\[2ex] c_2 \overset{\text{def}}{=} \sqrt{\dfrac{\sum n_i}{\sum n_i - n_2}} \; \dfrac{\Gamma(\frac{N+k-1}{2})}{\Gamma(\frac{N+k-2}{2})} \; \dfrac{1}{\sqrt{\pi}} , \\[2ex] c_1' \overset{\text{def}}{=} \sqrt{\dfrac{\sum n_i - n_2}{\sum n_i - n_1 - n_2}} \; \dfrac{\Gamma(\frac{N+k-2}{2})}{\Gamma(\frac{N+k-3}{2})} \; \dfrac{1}{\sqrt{\pi}} , \\[2ex] c_2' \overset{\text{def}}{=} \sqrt{\dfrac{\sum n_i - n_1}{\sum n_i - n_1 - n_2}} \; \dfrac{\Gamma(\frac{N+k-2}{2})}{\Gamma(\frac{N+k-3}{2})} \; \dfrac{1}{\sqrt{\pi}} . \end{cases}$$

Further we introduce the function $h_1(y)$ and $h_2(x)$, which are defined respectively for

$$-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}} \leqq y \leqq 0 \quad \text{and} \quad -\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_2}} \leqq x \leqq 0,$$

by the properties that respectively the points $\{h_1(y),y\}$ and $\{x,h_2(x)\}$ belong to the ellipse of figure 4.1.

Now we have

$$(4.35)\quad P\left[\underline{b}_1 \leqq g_1 \text{ and } \underline{b}_2 \leqq g_2\right] =$$

$$= c_1'c_2 \int_{h_2(g_1)}^{g_2} db_2 \int_{h_1(b_2)}^{g_1} \left(1-\frac{\sum n_i-n_2}{\sum n_i-n_1-n_2}b_1^2-\frac{2\sqrt{n_1n_2}}{\sum n_i-n_1-n_2}b_1b_2-\frac{\sum n_i-n_1}{\sum n_i-n_1-n_2}b_2^2\right)^{\frac{N+k-5}{2}} db_1$$

$$= c_1c_2' \int_{h_1(g_2)}^{g_1} db_1 \int_{h_2(b_1)}^{g_2} \left(1-\frac{\sum n_i-n_2}{\sum n_i-n_1-n_2}b_1^2-\frac{2\sqrt{n_1n_2}}{\sum n_i-n_1-n_2}b_1b_2-\frac{\sum n_i-n_1}{\sum n_i-n_1-n_2}b_2^2\right)^{\frac{N+k-5}{2}} db_2.$$

Applying the transformation (4.31) one finds

$$(4.36)\quad P\left[\underline{b}_1 \leqq g_1 \text{ and } \underline{b}_2 \leqq g_2\right] =$$

$$= c_1c_2' \int_{h_1(g_2)}^{g_1} db_1\left(1-\frac{\sum n_i}{\sum n_i-n_1}b_1^2\right)^{\frac{N+k-4}{2}} \int_{-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}}}^{\left(g_2+\frac{\sqrt{n_1n_2}}{\sum n_i-n_1}b_1\right)/\sqrt{1-\frac{\sum n_i}{\sum n_i-n_1}b_1^2}} \left(1-\frac{\sum n_i-n_1}{\sum n_i-n_1-n_2}(b_2')^2\right)^{\frac{N+k-5}{2}} db_2' .$$

In the same way, applying the transformation

$$(4.37)\qquad \underline{b}_1' = \frac{\underline{b}_1+\frac{\sqrt{n_1n_2}}{\sum n_i-n_2}\underline{b}_2}{\sqrt{1-\frac{\sum n_i}{\sum n_i-n_2}\underline{b}_2^2}} ,$$

it is found that

(4.38) $P\left[\underline{b}_1 \leqq g_1 \text{ and } \underline{b}_2 \leqq g_2\right] =$

$$= c_2 c_1' \int_{h_2(g_1)}^{g_2} db_2 \left(1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2\right)^{\frac{N+k-4}{2}} \int_{-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_2}}}^{\left(g_1 + \frac{\sqrt{n_1 n_2}}{\sum n_i - n_2} b_2\right) \Big/ \sqrt{1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2}} \left(1 - \frac{\sum n_i - n_2}{\sum n_i - n_1 - n_2} (b_1')^2\right)^{\frac{N+k-5}{2}} db_1' .$$

We have to prove

(4.39) $$\phi(g_1, g_2) \overset{\text{def}}{=} P\left[\underline{b}_1 \leqq g_1\right] \cdot P\left[\underline{b}_2 \leqq g_2\right] - P\left[\underline{b}_1 \leqq g_1 \text{ and } \underline{b}_2 \leqq g_2\right] > 0 .$$

First we have

(4.40) $$\phi\left(-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_2}},\ g_2\right) = P\left[\underline{b}_1 \leqq -\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_2}}\right] \cdot P\left[\underline{b}_2 \leqq g_2\right] - 0 > 0.$$

Now we consider (cf. 4.38)

(4.41) $$\phi(0, g_2) = \tfrac{1}{2} \cdot c_2 \int_{-\sqrt{\frac{\sum n_i - n_2}{\sum n_i}}}^{g_2} \left(1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2\right)^{\frac{N+k-4}{2}} db_2$$

$$-c_2 c_1' \int_{-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_1}}}^{g_2} db_2 \left(1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2\right)^{\frac{N+k-4}{2}} \int_{-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_2}}}^{\frac{\sqrt{n_1 n_2}}{\sum n_i - n_2} b_2 \Big/ \sqrt{1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2}} \left(1 - \frac{\sum n_i - n_2}{\sum n_i - n_1 - n_2} (b_1')^2\right)^{\frac{N+k-5}{2}} db_1'$$

$$\geqq \tfrac{1}{2} c_2 \int_{-\sqrt{\frac{\sum n_i - n_2}{\sum n_i}}}^{g_2} \left(1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2\right)^{\frac{N+k-4}{2}} db_2 +$$

$$- c_2 \int_{-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_1}}}^{g_2} \left(1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2\right)^{\frac{N+k-4}{2}} c_1' \int_{-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_2}}}^{\frac{\sqrt{n_1 n_2}}{\sum n_i - n_2} g_2 \Big/ \sqrt{1 - \frac{\sum n_i}{\sum n_i - n_2} g_2^2}} \left(1 - \frac{\sum n_i - n_2}{\sum n_i - n_1 - n_2} (b_1')^2\right)^{\frac{N+k-5}{2}} db_1'$$

for $\dfrac{\frac{\sqrt{n_1 n_2}\, b_2}{\sum n_i - n_2}}{\sqrt{1 - \frac{\sum n_i}{\sum n_i - n_2} b_2^2}}$ is monotonously increasing in b_2 .

Thus

$$(4.42)\quad \phi\left(0,-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}}\right)\geqq \tfrac{1}{2}c_2\int_{-\sqrt{\frac{\sum n_i-n_2}{\sum n_i}}}^{\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}}}\left(1-\frac{\sum n_i}{\sum n_i-n_2}b_2^2\right)^{\frac{N+k-4}{2}}>0 .$$

From (4.41) it follows that

$$(4.43)\quad \frac{d\,\phi(0,g_2)}{dg_2}=\tfrac{1}{2}c_2\left(1-\frac{\sum n_i}{\sum n_i-n_2}g_2{}^2\right)^{\frac{N+k-4}{2}}+$$

$$-\,c_2c_1'\left(1-\frac{\sum n_i}{\sum n_i-n_2}g_2^2\right)^{\frac{N+k-4}{2}}\int_{-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_2}}}^{\sqrt{\frac{n_1n_2}{\sum n_i-n_2}}g_2\Big/\sqrt{1-\frac{\sum n_i}{\sum n_i-n_2}g_2^2}}\left(1-\frac{\sum n_i-n_2}{\sum n_i-n_1-n_2}(b_1')^2\right)^{\frac{N+k-5}{2}}db_1'=$$

$$=c_2\left(1-\frac{\sum n_i}{\sum n_i-n_2}g_2^2\right)^{\frac{N+k-4}{2}}\phi_1(g_2),\ \text{say.}$$

Clearly $\phi_1(g_2)$ is a decreasing function of g_2 and as $\phi_1(0)=0$, we have

$$(4.44)\quad \frac{d\,\phi(0,g_2)}{dg_2}\geqq 0\quad\left(-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}}\leqq g_2\leqq 0\right).$$

From (4.42) and (4.44) it follows that

$$(4.45)\quad \phi(0,g_2)>0\quad\left(-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}}\leqq g_2\leqq 0\right).$$

Next we consider (cf. 4.36)

$$(4.46)\quad \frac{\partial\,\phi(g_1,g_2)}{\partial g_1}=c_1\left(1-\frac{\sum n_i}{\sum n_i-n_1}g_1^2\right)^{\frac{N+k-4}{2}}c_2\int_{-\sqrt{\frac{\sum n_i-n_2}{\sum n_i}}}^{g_2}\left(1-\frac{\sum n_i}{\sum n_i-n_2}b_2^2\right)^{\frac{N+k-4}{2}}db_2+$$

$$-\,c_1c_2'\left(1-\frac{\sum n_i}{\sum n_i-n_1}g_1^2\right)^{\frac{N+k-4}{2}}\int_{-\sqrt{\frac{\sum n_i-n_1-n_2}{\sum n_i-n_1}}}^{\left(g_2+\sqrt{\frac{n_1n_2}{\sum n_i-n_1}}g_1\right)\Big/\sqrt{1-\frac{\sum n_i}{\sum n_i-n_1}g_1^2}}\left(1-\frac{\sum n_i-n_1}{\sum n_i-n_1-n_2}(b_2')^2\right)^{\frac{N+k-5}{2}}db_2'=$$

$$=c_1\left(1-\frac{\sum n_i}{\sum n_i-n_1}g_1^2\right)^{\frac{N+k-4}{2}}\cdot\phi_2(g_1,g_2),\ \text{say.}$$

The partial derivative with respect to g_1 of the upper bound of the second integral of $\phi_2(g_1,g_2)$ is

$$(4.47)\qquad \frac{\frac{\sqrt{n_1n_2}}{\sum n_i - n_1} + g_1g_2\cdot\frac{\sum n_i}{\sum n_i - n_1}}{\left(1-\frac{\sum n_i}{\sum n_i - n_1}g_1^2\right)^{\frac{3}{2}}} > 0, \text{ if } g_1g_2 \geqq 0,$$

thus $\phi_2(g_1,g_2)$ is a decreasing function of g_1 in the domain under consideration. Further $\left(1-\frac{\sum n_i}{\sum n_i - n_1}g_1^2\right)^{\frac{N+k-4}{2}}$ is positive. Thus $\frac{\partial\phi(g_1,g_2)}{\partial g_1}$ is everywhere negative, everywhere positive, or positive up to a certain point g_o (depending upon g_2), say, and negative thereafter. So in virtue of (4.40) and (4.45) we may conclude

$$(4.48)\qquad \phi(g_1,g_2) > 0,\ \left(-\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_2}} \leqq g_1 \leqq 0,\ -\sqrt{\frac{\sum n_i - n_1 - n_2}{\sum n_i - n_1}} \leqq g_2 \leqq 0\right).$$

5. Slippage tests for some discrete variables

In this section slippage tests will be discussed for variates which follow the Poisson, the binomial or the negative binomial law. First we shall consider the Poisson case in some detail. Suppose we have a set of independent random variables

$$(5.1)\qquad \underline{z}_1,\ldots,\underline{z}_k,$$

distributed according to Poisson distributions, i.e.:

$$(5.2)\qquad P\left[\underline{z}_i = z_i\right] = \frac{e^{-\mu_i}\mu_i^{z_i}}{z_i!},\ (i = 1,\ldots,k),\ \mu_i > 0\ .$$

Now we want to test the hypothesis H_o that the means μ_i have known ratios

$$(5.3)\qquad H_o:\ \frac{\mu_i}{\sum_j \mu_j} = p_i \quad (i = 1,\ldots,k).$$

This situation occurs for instance if from k Poisson-populations with, under H_o, equal means unequal numbers of observations are present and $\underline{z}_1,\ldots,\underline{z}_k$ represent the sums of these observations. In this case the p_i are proportional to the numbers of observations. Also k Poisson processes with the same parameter may be observed during different lengths of time. Then the p_i are proportional to these lengths of time.

We want to test H_o against the alternatives

$$(5.4)\qquad H_1:\ \frac{\mu_i}{\sum_j \mu_j} = cp_i,\ \frac{\mu_l}{\sum_j \mu_j} = \frac{1-cp_i}{1-p_i}\cdot p_l\ (l \neq i),\ 1 < c < \frac{1}{p_i}\ ,\ c \text{ unknown},$$

for one unknown value of i or

(5.5) $\quad H_2: \frac{\mu_i}{\sum_j \mu_j} = cp_i, \; \frac{\mu_l}{\sum_j \mu_j} = \frac{1-cp_i}{1-p_i} p_l \; (l \neq i), \; 0 < c < 1, \; c$ unknown,

for one unknown value of i.

A well known property of Poisson-variates is: If $\underline{z}_1, \ldots, \underline{z}_k$ are independent Poisson-variates with means $\mu_1, \ldots, \mu_k$, then the simultaneous conditional distribution of $\underline{z}_1, \ldots, \underline{z}_k$ given their sum (i.e. $\sum \underline{z}_i = N$, N a constant), is a multinomial distribution with probabilities $p_i = \frac{\mu_i}{\sum_j \mu_j}$ and number of trials $\sum \underline{z}_i = N$. As the hypotheses (5.3), (5.4) and (5.5) only contain the ratios p_i it seems natural to use a conditional test for H_o, using only the multinomial distribution

(5.6) $P\left[\underline{z}_1 = z_1, \ldots, \underline{z}_k = z_k \,\middle|\, \sum \underline{z}_i = N\right] = \frac{N!}{\prod z_i!} \prod p_i^{\,z_i}$, if $\sum z_i = N$ and 0 otherwise.

From this it is clear that a test against slippage for Poisson variates is closely related to a similar test for a multinomial distribution. The reader may easily translate the tests stated here into tests for the multinomial case.

In the next section the following theorem will be proved.

Theorem 5.1. Suppose the discrete, random variables

(5.7) $$\underline{u}_1, \ldots, \underline{u}_k$$

are distributed independently and can take integer values only (the latter assumption is not essential but gives a much simpler notation).

If

(5.8) $$\frac{P\left[\sum \underline{u}_l - \underline{u}_i - \underline{u}_j = a\right]}{P\left[\sum \underline{u}_l - \underline{u}_i - \underline{u}_j = a+1\right]},$$

where a is an integer, is a non decreasing function of a, then

(5.9) $P\left[\underline{u}_i \geq u_i \text{ and } \underline{u}_j \geq u_j \,\middle|\, \sum \underline{u}_l = N\right] \leq P\left[\underline{u}_i \geq u_i \,\middle|\, \sum \underline{u}_l = N\right] . P\left[\underline{u}_j \geq u_j \,\middle|\, \sum \underline{u}_l = N\right]$,

for every pair of integers u_j and u_i and for every non-negative integer N.

In the special case where $\underline{u}_1, \ldots, \underline{u}_k$ are distributed according to the same type of distribution and this distribution has the property that a sum of k independent variates has again the same type of distribution, it is easy to verify whether condition (5.8) holds or not.

In our case the sum of (k-2) of the variables $\underline{z}_i$ (given by 5.2) has a Poisson-distribution with mean μ, say. So condition (5.8) reads

(5.10) $$\frac{e^{-\mu} \mu^a}{a!} \cdot \frac{(a+1)!}{e^{-\mu} \mu^{a+1}} = \frac{a+1}{\mu},$$

is non decreasing in a, which is clearly true.

Thus the inequality (5.9) holds for every pair $\underline{z}_i, \underline{z}_j$ and the procedure described in section 2 may be applied to the variables $\underline{z}_1, \ldots, \underline{z}_k$, under the condition $\sum \underline{z}_i = N$.[1] Now the marginal distribution of $\underline{z}_i$ under the condition $\sum \underline{z}_i = N$ is a binomial one, so when testing H_o against H_1 we compute, if $z_1, \ldots, z_k$ are the observed values and $\sum z_i = N$

$$(5.11) \quad r_i \stackrel{\text{def}}{=} P\left[\underline{z}_i \geq z_i \,\middle|\, \sum \underline{z}_1 = N\right] = \sum_{x=z_i}^{N} \binom{N}{x} p_i^{\,x} (1-p_i)^{N-x} = I_{p_i}(z_i, N-z_i+1) \;.$$

Now H_o is rejected if

$$(5.12) \qquad \min r_i \leq \frac{\varepsilon}{k}$$

and then we decide that $\frac{\mu_j}{\sum \mu_i} > p_j$ if j is the smallest integer for which $r_j = \min r_i$.

If under H_o $\mu_1 = \ldots = \mu_k$, all p_i are equal and the smallest r_i corresponds to the largest value z_i.

The test for slippage to the left is completely analogous.

A table of critical values for max z_i is given in section 11 for the case $p_1 = p_2 = \ldots = p_k$.

Along the same lines as was done by R. DOORNBOS and H.J. PRINS (1956) in the case of Γ-variates it can be shown that the probability Q_j of making the correct decision when the j^{th} population has slipped to the right (i.e. H_1 is true with $i = j$) satisfies the inequality

$$(5.13) \quad I_{cp_j}(G_{j,\varepsilon}, N-G_{j,\varepsilon}+1)\left[1-\sum_{i\neq j} I_{\frac{1-cp_j}{1-p_j}p_i}(G_{i,\varepsilon}, N-G_{i,\varepsilon}+1)\right] \leq Q_j \leq I_{cp_j}(G_{j,\varepsilon}, N-G_{j,\varepsilon}+1).$$

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1) The validity of (5.9) in the case of Poisson-variates can also be proved in the following way, using the relation with Γ-variates. The well known relation

$$(1) \quad P\left[\underline{z}_1 \geq z_1 \,\middle|\, \sum \underline{z}_i = N\right] = \sum_{x=z_1}^{N} \binom{N}{x} p_1^{\,x}(1-p_1)^{N-x} = \frac{N!}{(z_1-1)!(N-z_1)!}\int_0^{p_1} u^{z_1-1}(1-u)^{N-z_1}du$$

can be generalized to

$$(2) \quad \sum_{x_1=z_{i_1}}^{N} \cdots \sum_{x_r=z_{i_r}}^{N} \frac{N!}{x_1!\ldots x_r!(N-x_1-\ldots-x_r)!}\, p_{i_1}^{x_1}\ldots p_{i_r}^{x_r}(1-p_{i_1}\ldots-p_{i_r})^{N-x_1,\ldots-x_r} =$$

$$= \frac{N!}{(z_{i_1}-1)!\ldots(z_{i_r}-1)!(N-z_{i_1}-\ldots-z_{i_r})!}\int_0^{p_{i_1}}\!\!\ldots\!\int_0^{p_{i_r}} u_1^{z_{i_1}-1}\ldots u_r^{z_{i_r}-1}(1-u_1\ldots-u_r)^{N-z_{i_1}\ldots-z_{i_r}}\,du_1\ldots du_r,$$

$$(r \leq k-1, (i_1, \ldots, i_r) \in (1, \ldots, k)).$$

which may be proved by induction or otherwise. Using (2) for r=2 it is seen immediately that inequality (4.10) in R. DOORNBOS and H.J. PRINS (1956) is the same as (5.9) for Poisson variates.

Here $G_{l,\varepsilon}$ $(l = 1,\dots,k)$ is the smallest number which satisfies

$$(5.14) \qquad P\left[\underline{z}_l \geq G_{l,\varepsilon} \middle| \Sigma \underline{z}_i = N,\ H_o\right] \leq \varepsilon/k,$$

or

$$(5.15) \qquad I_{p_l}(G_{l,\varepsilon},\ N-G_{l,\varepsilon}+1) \leq \varepsilon/k.$$

Clearly Q_j converges towards its upper bound when $c \rightarrow 1/p_j$ and for each $c \geq 1$ the factor between square brackets is larger than $1 - \frac{k-1}{k}\,\varepsilon$, according to (5.15).

In the case of slippage to the left we have analogously

$$(5.16) \qquad \left[1-I_{cp_j}(g_{j,\varepsilon},\ N-g_{j,\varepsilon}+1)\right](1-\varepsilon) \leq$$

$$\left[1-I_{cp_j}(g_{j,\varepsilon},\ N-g_{j,\varepsilon}+1)\right]\left[1-\sum_{i\neq j}\left\{1-I_{\frac{1-cp_j}{1-p_j}p_i}(g_{i,\varepsilon},\ N-g_{i,\varepsilon}+1)\right\}\right]$$

$$\leq P_j \leq 1-I_{cp_j}(g_{j,\varepsilon},\ N-g_{j,\varepsilon}+1),$$

where $g_{l,\varepsilon}$ $(l = 1,\dots,k)$ is the largest number satisfying

$$(5.17) \qquad 1 - I_{p_l}(g_{l,\varepsilon}+1,\ N-g_{l,\varepsilon}) \leq \frac{\varepsilon}{k}.$$

We can apply theorem (5.1) also to the case of independent variables

$$(5.18) \qquad \underline{v}_1,\dots,\underline{v}_k,$$

which are distributed according to binomial laws with numbers of trials $n_1,\dots,n_k$ and probabilities of success $p_1,\dots,p_k$. Now the hypothesis H_o is

$$(5.19) \qquad H_o:\ p_1 = \dots = p_k = p, \text{ say}$$

and the alternatives are

$$(5.20) \qquad \begin{aligned} H_1:\ p_1 &= p_2 \dots = p_{i-1} = p_{i+1} = \dots = p_k = p, \\ p_i &= cp \quad (1 \leq c \leq 1/p), \end{aligned}$$

for one unknown value of i and

$$(5.21) \qquad \begin{aligned} H_2:\ p_1 &= \dots = p_{i-1} = p_{i+1} = \dots = p_k = p, \\ p_i &= cp \quad (0 \leq c \leq 1), \end{aligned}$$

for one unknown value of i.

Because, under H_o, the sum of $(k-2)$ of the variates (5.18) has again a binomial distribution with number of trials, n say, and probability of a success in each trial p, the condition (5.8) of theorem 5.1 reads

$$(5.22) \qquad \frac{\binom{n}{a}p^a(1-p)^{n-a}}{\binom{n}{a+1}p^{a+1}(1-p)^{n-a-1}} = \frac{a+1}{n-a}\cdot\frac{1-p}{p}$$

is a non decreasing function of a, which is true. So in this case also the approximation procedure described in section 2 can be applied to obtain a conditional test for slippage under the condition that the sum of the variates $\Sigma \underline{v}_i$ has a constant value N. The conditional distribution of $\underline{v}_i$ is a hypergeometrical one

$$(5.23) \qquad P\left[\underline{v}_i = v_i \middle| \Sigma \underline{v}_i = N\right] = \frac{\binom{n_i}{v_i}\binom{\Sigma n_j - n_i}{N - v_i}}{\binom{\Sigma n_j}{N}}, \quad (\underline{v}_i \geqq 0),$$

so with help of this distribution critical values for the tests with prescribed level of significance may be obtained, in the same way as was done with the Poisson variates.

Provided that none of the values n_i, $\Sigma n_j - n_i$, N and $\Sigma n_j - N$ are very small, a good approximation to the sum of the tail terms of the hypergeometric series of equation (5.23) may be obtained from the integral under a normal curve, having the mean $\frac{n_i . N}{\Sigma n_j}$ and variance

$$\frac{n_i(\Sigma n_j - n_i)N(\Sigma n_j - N)}{(\Sigma n_j)^2(\Sigma n_j - 1)} .$$

In the special case $n_1 = \ldots = n_k = n$, the test procedure for slippage to the right reduces to comparing the largest variate $\underline{v}_m$ with a constant v_o determined by the level of significance ε, such that v_o is the largest value satisfying

$$P\left[\underline{v}_i \geqq v_o \middle| \Sigma \underline{v}_i = N\right] \leqq \varepsilon/k .$$

The same holds for the variates

$$(5.24) \qquad \underline{w}_1, \ldots, \underline{w}_k ,$$

which are independently distributed according to <u>negative binomial</u> laws, with parameters $r_1, \ldots, r_k$ and probabilities $p_1, \ldots, p_k$, i.e.

$$(5.25) \qquad P\left[\underline{w}_i = w_i\right] = \binom{w_i + r_i - 1}{r_i - 1} p_i^{r_i} q_i^{w_i} ,$$

where r_i is an integer $\geqq 1$ and $0 \leqq p_i \leqq 1$, whilst $p_i + q_i = 1$.

The hypothesis H_o is

$$(5.26) \qquad H_o: \quad q_1 = \ldots = q_k = q, \text{ say}$$

and the alternatives are

$$(5.27) \qquad H_1: \quad q_1 = \ldots = q_{i-1} = q_{i+1} = \ldots = q_k = q,$$
$$q_i = cq \quad (1 \leqq c \leqq 1/q),$$

for one unknown value of i or

$$(5.28)\qquad H_2:\ q_1 = \dots = q_{i-1} = q_{i+1} = \dots = q_k = q\ ,\qquad q_i = cq\quad (0 \leqq c \leqq 1),$$

for one unknown value of i.

The hypotheses are stated in terms of the q_i and not in terms of the p_i in order to obtain that slippage to the right of the i^{th} population corresponds to a large value of $\underline{w}_i$.

Under H_o, the sum of a set of independent negative binomial variates has again a negative binomial distribution with the same probability p (or q) and a parameter r, say, which is the sum of the r_i of the individual variates. So condition (5.8) gives here

$$(5.29)\qquad \frac{\binom{a+r-1}{r-1}p^r q^a}{\binom{a+r}{r-1}p^r q^{a+1}} = \frac{a+1}{a+r}\cdot\frac{1}{q}\ ,$$

is a non decreasing function of a, which is true if $r \geqq 1$. Thus again the method of section 2 may be applied. The conditional distribution of $\underline{w}_i$ under the condition $\Sigma \underline{w}_j = N$, has the form

$$(5.30)\quad P\left[\underline{w}_i = w_i \middle| \Sigma \underline{w}_j = N\right] = \frac{\binom{w_i+r_i-1}{r_i-1}\binom{N+\Sigma r_j-w_i-r_i-1}{\Sigma r_j-r_i-1}}{\binom{N+\Sigma r_j-1}{\Sigma r_j-1}}\ ,\qquad (w_i = 0,1,\dots,N).$$

The critical region for the test against H_1 (5.27) consists of large values of the variables $\underline{w}_i$. In the case where $r_1 = \dots = r_k$ the test statistic is the largest variate $\underline{w}_m$, when testing against slippage to the right and the smallest when testing against slippage to the left.

If in the case of the variables (5.1), (5.18) and (5.24) holds that $p_1 = \dots = p_k$, $n_1 = \dots = n_k$ and $r_1 = \dots = r_k$ respectively, then in each case the following optimum property can be proved.[1] As in the case of the normal distribution we denote by D_o the decision that H_o is true and by D_i $(i = 1,\dots,k)$ the decision that H_{1i} is true, i.e. that H_1 is true and that the i^{th} population has slipped to the right. Now the procedure:

$$(5.31)\qquad \begin{cases} \text{if } \underline{u}_m > \lambda_{\varepsilon,N} \text{ select } D_m, \\ \text{if } \underline{u}_m \leqq \lambda_{\varepsilon,N} \text{ select } D_o, \end{cases}$$

under the condition that $\Sigma \underline{u}_i = N$ where $\underline{u}$ stands for $\underline{z}$, $\underline{v}$, $\underline{w}$ according as the Poisson, the binomial or the negative binomial case is concerned and where m is the index of the maximum $\underline{u}$-value, maximizes the probabi-

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1) In the sequel only the case of slippage to the right is considered but all statements may be easily translated for the other case.

lity of making a correct decision when H_1 is true subject to the following restrictions:

(a) When H_o is true, D_o should be selected with probability $\geqq 1-\varepsilon$,

(b) The probability of making a correct decision when the i-th population has slipped by an amount c must be the same for $i = 1,\dots,k$.

The constant $\lambda_{\varepsilon,N}$ in (5.31) is determined by the level of significance ε and depends on N, the sum of the variables.

In the binomial and the negative binomial case this optimum property follows from

Theorem 5.2. Suppose the discrete, random variables

$$\underline{x}_1,\dots,\underline{x}_k$$

are under H_o distributed independently according to the same distribution function, then for each value of N, the procedure (5.31) is optimum in the abovementioned sense if

$$\frac{P[\underline{x}_i=x\,|H_{1i}]}{P[\underline{x}_i=x\,|H_o]}\ , \tag{5.32}$$

is a non decreasing function of x for every c.[1)]

This theorem will be proved in section 6.

Applying it to the two distributions under consideration we get in case of the binomial and the negative binomial distribution the conditions that respectively

$$\frac{\binom{n}{x}(cp)^x(1-cp)^{n-x}}{\binom{n}{x}\,p^x\,(1-p)^{n-x}} = \left(\frac{c-cp}{1-cp}\right)^x\left(\frac{1-cp}{1-p}\right)^n, \qquad (c>1) \tag{5.33}$$

and

$$\frac{\binom{x+r-1}{r-1}(1-cq)^r(cq)^x}{\binom{x+r-1}{r-1}(1-q)^r\,q^x} = \left(\frac{1-cq}{1-q}\right)^r c^x, \qquad (c>1) \tag{5.34}$$

are non decreasing functions of x, which is true in both cases.

For the Poisson distribution a separate proof will be given in section 6.

6. Proofs of the results stated in section 5 and a general condition for the inequality (2.9) in the continuous case.

Starting with the proof of theorem 5.1 we have that

$$\frac{P[\underline{u}_i=y].P[\underline{u}_j=x].P[\sum \underline{u}_l-\underline{u}_i-\underline{u}_j=N-x-y]}{P[\underline{u}_i=y].P[\underline{u}_j=x+1].P[\sum \underline{u}_l-\underline{u}_i-\underline{u}_j=N-x-y-1]} \tag{6.1}$$

---

1) In case of slippage to the left (5.32) should be non increasing.

is non decreasing in y, according to (5.8). Dividing (6.1) by the factor

$$(6.2)\qquad \frac{P\left[\sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right]}{P\left[\sum \underline{u}_l=N \text{ and } \underline{u}_j=x+1\right]}\,,$$

which does not depend on y, (6.1) changes into

$$(6.3)\qquad \frac{P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right]}{P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x+1\right]}\,.$$

Thus also (6.3) is non decreasing in y for all values of x. This means that there exists a value y_o, which may depend on x, which has the property that

$$(6.4)\qquad P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right] \geqq P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x+1\right]\,, \text{ if } y \geqq y_o\,,$$

$$P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right] \leqq P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x+1\right]\,, \text{ if } y < y_o\,.$$

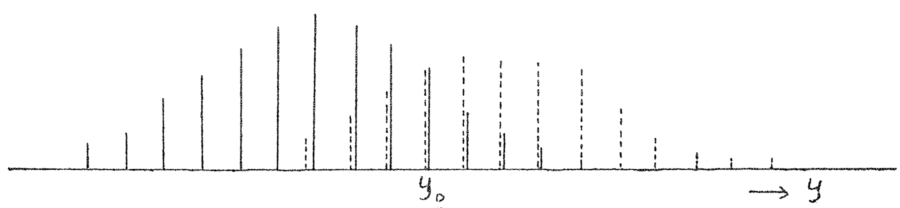

fig. 6.1

$P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right]$ (dotted lines),

and

$P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x+1\right]$ (full lines).

This situation is sketched in figure 6.1. It follows that for each value u_i

$$(6.5)\qquad P(x) \overset{\text{def}}{=} \sum_{y=u_i}^{\infty} P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right]$$

is a non increasing function of x. Now

$$(6.6)\qquad \frac{P\left[\underline{u}_i \geqq u_i \text{ and } \underline{u}_j \geqq u_j \,\middle|\, \sum \underline{u}_l=N\right]}{P\left[\underline{u}_j \geqq u_j \,\middle|\, \sum \underline{u}_l=N\right]} =$$

$$= \frac{\sum_{x=u_j}^{\infty} P\left[\underline{u}_j=x \,\middle|\, \sum \underline{u}_l=N\right] \sum_{y=u_i}^{\infty} P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=x\right]}{\sum_{x=u_j}^{\infty} P\left[\underline{u}_j=x \,\middle|\, \sum \underline{u}_l=N\right]}$$

$$\leqq \sum_{y=u_i}^{\infty} P\left[\underline{u}_i=y \,\middle|\, \sum \underline{u}_l=N \text{ and } \underline{u}_j=u_j\right]\,.$$

In the same way we have

$$(6.7)\quad \frac{P\left[\underline{u}_i \geq u_i \text{ and } \underline{u}_j < u_j \,\middle|\, \sum \underline{u}_l = N\right]}{P\left[\underline{u}_j < u_j \,\middle|\, \sum \underline{u}_l = N\right]} \geq \sum_{y=u_i}^{\infty} P\left[\underline{u}_i = y \,\middle|\, \sum \underline{u}_l = N \text{ and } \underline{u}_j = u_j\right].$$

From (6.6) and (6.7) it follows that, in the notation of (2.6), where $u_i = g_i + 1$ and $u_j = g_j + 1$, whilst $\underline{u}_i$ under the condition $\sum \underline{u}_l = N$ stands for $\underline{x}_i$ and $\underline{u}_j$ under the condition $\sum \underline{u}_l = N$ for $\underline{x}_j$,

$$(6.8)\qquad \frac{q_{i,j}}{q_j} \leq \frac{q_i - q_{i,j}}{1 - q_j},$$

or

$$(6.9)\qquad q_{i,j} \leqq q_i q_j,$$

which proves the theorem, because (6.9) is the same as (5.9).

Following a somewhat similar line of thought in the continuous case we arrive at the following theorem:

<u>Theorem 6.1. Suppose the random variables $\underline{x}$ and $\underline{y}$ have a joint distribution, which is given by the density function $f(x,y)$. Now the inequality</u>

$$(6.10)\qquad P\left[\underline{x} \leqq a \text{ and } \underline{y} \leqq b\right] \leq P\left[\underline{x} \leqq a\right] . P\left[\underline{y} \leqq b\right],$$

<u>holds for all real values a and b, if</u>

$$(6.11)\quad f(x_1,y_1)f(x_2,y_2) \leqq f(x_2,y_1)f(x_1,y_2), \ \underline{\text{for}}\ x_1 \leqq x_2 \text{ and } y_1 \leqq y_2 .$$

<u>Proof</u>: From (6.11) it follows that

$$(6.12)\quad \int_{x_1=-\infty}^{a} \int_{y_1=-\infty}^{b} \int_{x_2=a}^{\infty} \int_{y_2=b}^{\infty} \left[f(x_1,y_1)f(x_2,y_2) - f(x_2,y_1)f(x_1,y_2)\right] dx_1 dy_1 dx_2 dy_2 \leqq 0.$$

Or

$$(6.13)\quad \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} f(x,y)\,dxdy \int_{x=a}^{\infty} \int_{y=b}^{\infty} f(x,y)\,dxdy$$
$$\leqq \int_{x=a}^{\infty} \int_{y=-\infty}^{b} f(x,y)\,dxdy \int_{x=-\infty}^{a} \int_{y=b}^{\infty} f(x,y)\,dxdy .$$

Adding to both sides of (6.13) the product

$$(6.14)\quad \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} f(x,y)\,dxdy \int_{x=-\infty}^{a} \int_{y=b}^{\infty} f(x,y)\,dxdy,$$

(6.13) passes into

$$(6.15)\quad \int_{x=-\infty}^{a}\int_{y=-\infty}^{b} f(x,y)\,dxdy \int_{x=-\infty}^{\infty}\int_{y=b}^{\infty} f(x,y)\,dxdy$$

$$\leqq \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{b} f(x,y)\,dxdy \int_{x=-\infty}^{a}\int_{y=b}^{\infty} f(x,y)\,dxdy ,$$

or

$$(6.16)\quad \frac{\int_{x=-\infty}^{a}\int_{y=-\infty}^{b} f(x,y)\,dxdy}{\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{b} f(x,y)\,dxdy} \leqq \frac{\int_{x=-\infty}^{a}\int_{y=b}^{\infty} f(x,y)\,dxdy}{\int_{x=-\infty}^{\infty}\int_{y=b}^{\infty} f(x,y)\,dxdy} ,$$

or

$$(6.17)\qquad P\left[\underline{x} \leqq a \mid \underline{y} \leqq b\right] \leqq P\left[\underline{x} \leqq a \mid \underline{y} > b\right] ,$$

which is equivalent to (6.10) (cf. the transition from (6.8) to (6.9)).

Remark

The condition (6.11) is certainly satisfied in the special case where $\frac{\partial^2 \log f(x,y)}{\partial x \partial y}$ exists everywhere and is everywhere non positive. For (6.11) says

$$(6.18)\qquad \frac{f(x_1,y_1)}{f(x_2,y_1)} \leqq \frac{f(x_1,y_2)}{f(x_2,y_2)}$$

if $x_1 \leqq x_2$ and $y_1 \leqq y_2$.

(6.18) holds if $\frac{\partial}{\partial y}\ \frac{f(x_1,y)}{f(x_2,y)} \geqq 0$ if $x_1 \leqq x_2$

or

$$(6.19)\quad \partial_y f(x_1,y)\,.\,f(x_2,y) - f(x_1,y)\partial_y f(x_2,y) \geqq 0 \qquad \text{if } x_1 \leqq x_2 .$$

The inequality (6.19) may be written

$$(6.20)\qquad \frac{\partial \log f(x_1,y)}{\partial y} \geqq \frac{\partial \log f(x_2,y)}{\partial y} \qquad \text{if } x_1 \leqq x_2 ,$$

which is certainly satisfied

if $\frac{\partial^2 \log f(x\ ,y)}{\partial x\,\partial y} \leqq 0$ everywhere.

If $f(x_1,y_1)f(x_2,y_2) \geqq f(x_2,y_1)f(x_1,y_2)$

everywhere, where $x_1 \leqq x_2$ and $y_1 \leqq y_2$

then $P\left[\underline{x} \leqq a \text{ and } \underline{y} \leqq b\right] \geqq P\left[\underline{x} \leqq a\right] . P\left[\underline{y} \leqq b\right]$

instead of (6.10).

Theorem 6.1 does not seem to have many practical applications. As an example we may consider the bivariate normal distribution, where the density function has the form

$$(6.21)\quad f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\frac{x_1-\mu_1}{\sigma_1}\cdot\frac{x_2-\mu_2}{\sigma_2}+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}.$$

Here we have

$$(6.22)\qquad \frac{\partial^2}{\partial x_1\partial x_2}\log f(x_1,x_2) = \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} ,$$

thus the inequality (6.10) holds if the correlation coefficient ρ is negative. This case of the inequality (6.10) was recently used by H.A. DAVID (1956) but no proof was given.

The proof of theorem 5.2 follows the lines indicated by E. PAULSON (1952) and D.R. TRUAX (1953). It consists mainly in showing that for any c, N and p or q there exists a set of non zero a priori probabilities $g_0,g_1,\ldots,g_k$, which are functions of N andp or q so that, when g_i is the probability that D_i is the correct decision the decision procedure described in section 5 maximizes the probability of making the correct decision. Assuming this has been demonstrated, it follows easily that (5.31) is the optimum solution. For suppose there existed an allowable decision procedure, which for some c and N and p or q had a greater probability than (5.31) of making the correct decision when some category had slipped to the right by an amount c. Then this procedure will have a greater probability than (5.31) of making a correct decision (for that values of c, N and p or q) with respect to any set of a priori probabilities, with $\max_{1\le i\le k} g_i > 0$, which would be a contradiction.

According to A. WALD (1950), pp 127-128 the optimum solution is given by the rule:"For each j (j= 0,1,...,k) decide D_j for all points in the sample space where j is the smallest integer for which g_jf_j = $\max\left\{g_0f_0,g_1f_1,\ldots,g_kf_k\right\}$, where f_j is the joint elementary probability law of $\underline{x}_1,\ldots,\underline{x}_k$ under the hypothesis H_{1j}."

We consider the special a priori distribution $g_0=1-kg$, $g_1 = \ldots = g_k = g$. For example the region where D_1 is selected is given by the points in the sample space where $f_1>f_i$ $(i = 2,\ldots,k)$ and $gf_1>(1-kg)f_0$. The region where $f_1>f_i$ is given by

$$(6.23)\quad \frac{P\left[\underline{x}_1=x_1\mid H_{11}\right]\cdots P\left[\underline{x}_k=x_k\mid H_{11}\right]}{P\left[\sum\underline{x}_i=N\mid H_{11}\right]}>\frac{P\left[\underline{x}_1=x_1\mid H_{1i}\right]\cdots P\left[\underline{x}_k=x_k\mid H_{1i}\right]}{P\left[\sum\underline{x}_i=N\mid H_{1i}\right]},$$

$$(\textstyle\sum\underline{x}_i=N).$$

Because $\underline{x}_1,\ldots,\underline{x}_k$ have the same distribution and on account of the form of the hypotheses H_{1i} we have

$$(6.24)\begin{cases} P\left[\sum \underline{x}_i = N \mid H_{1j}\right] \text{ is the same for } j = 1,\ldots,k\,, \\ P\left[\underline{x}_i = x \mid H_{1j}\right] = P\left[\underline{x}_i = x \mid H_o\right] \text{ for } j = 1,\ldots,k;\ j \neq i, \\ P\left[\underline{x}_i = x \mid H_{1i}\right] = P\left[\underline{x}_j = x \mid H_{1j}\right],\ \text{for } i,j = 1,\ldots,k, \\ P\left[\underline{x}_i = x \mid H_o\right] = P\left[\underline{x}_j = x \mid H_o\right],\ \text{for } i,j = 1,\ldots,k. \end{cases}$$

With help of these relations (6.23) reduces to

$$(6.25)\qquad \frac{P\left[\underline{x}_i = x_1 \mid H_{1i}\right]}{P\left[\underline{x}_i = x_1 \mid H_o\right]} > \frac{P\left[\underline{x}_i = x_i \mid H_{1i}\right]}{P\left[\underline{x}_i = x_i \mid H_o\right]}\,,$$

which is equivalent to $x_1 > x_i$ on account of the condition (5.32) of the theorem.

The region where $gf_1 > (1-kg)f_o$ is given by

$$(6.26)\quad g\frac{P\left[\underline{x}_1 = x_1 \mid H_{11}\right]\ldots P\left[\underline{x}_k = x_k \mid H_{11}\right]}{P\left[\sum \underline{x}_i = N \mid H_{11}\right]} > (1-kg)\frac{P\left[\underline{x}_1 = x_1 \mid H_o\right]\ldots P\left[\underline{x}_k = x_k \mid H_o\right]}{P\left[\sum \underline{x}_i = N \mid H_o\right]}\,,$$

or, on account of (6.24) by

$$(6.27)\qquad \frac{P\left[\underline{x}_1 = x_1 \mid H_{11}\right]}{P\left[\underline{x}_1 = x_1 \mid H_o\right]} > \frac{1-kg}{g}\,\frac{P\left[\sum \underline{x}_i = N \mid H_{11}\right]}{P\left[\sum \underline{x}_i = N \mid H_o\right]}\,.$$

In virtue of (5.32) this is equivalent to $x_1 > L$, where L is a number depending on N, and p or q (L may be $+\infty$). Thus the Bayes solution is: if x_m is the maximum of $x_1,\ldots,x_k$ select D_m if $x_m > L$, otherwise select D_o. Define the function $F(g)$ by the equation

$$(6.28)\quad F(g) = \frac{P\left[\underline{x}_1 = \lambda_{\varepsilon,N} \mid H_{11}\right]}{P\left[\underline{x}_1 = \lambda_{\varepsilon,N} \mid H_o\right]} - \frac{1-kg}{g}\,\frac{P\left[\sum \underline{x}_i = N \mid H_{11}\right]}{P\left[\sum \underline{x}_i = N \mid H_o\right]}\,,$$

where $\lambda_{\varepsilon,N}$ is the constant used in (5.31). It is obvious that $F(g)$ is a continuous function of g, with $F(\frac{1}{k}) > 0$ and that there exists a δ with $0 < \delta < \frac{1}{k}$ such that $F(\delta) < 0$. Hence there exists a value g^* with $0 < \delta < g^* < \frac{1}{k}$ such that $F(g^*) = 0$. To get the Bayes solution relative to $(1-kg^*, g^*,\ldots,g^*)$ it is only necessary in the solution given above to replace L by $\lambda_{\varepsilon,N}$. Thus the procedure (5.31) is the Bayes solution relative to $(1-kg^*, g^*,\ldots,g^*)$, which proves that it is an optimum one.

In the case of the Poisson variates (5.1), with under H_o (5.3) $p_1 = \ldots = p_k = \frac{1}{k}$, we start directly from their joint distribution as given by (5.6), which reads in this special case:

$$(6.29)\quad \begin{cases} f_o(z_1,\ldots,z_k) = \dfrac{N!}{\prod z_i!}\left(\dfrac{1}{k}\right)^N, \\ f_i(z_1,\ldots,z_k) = \dfrac{N!}{\prod z_i!}\left(\dfrac{1}{k}\right)^N c^{z_i}\left(\dfrac{k-c}{k-1}\right)^{N-z_i} \quad (1<c<k). \end{cases}$$

Because

$$(6.30)\qquad c^{z_i}\left(\frac{k-c}{k-1}\right)^{N-z_i},$$

is monotonously increasing in z_i for $1<c<k$, WALD's rule may be applied in the same way as was done in the preceding proof as also here the region where $f_1>f_i$ is given by $z_1>z_i$ and the region where $gf_1>(1-kg)f_o$ by $z_1>L$, L depending on N and c.

7. Slippage tests for the method of m rankings

In the well known method of m rankings due to M. FRIEDMAN (1937) (cf. M.G. KENDALL (1955), chapters 6 and 7) m "observers" are considered. Each observer ranks k "objects". The method of m rankings enables us to investigate whether the observers agree in their opinion about the objects. For that reason one tests the hypothesis H_o, which states that the rankings are chosen at random from the collection of all permutations of the numbers $1,\ldots,k$ and that they are independent.

Here we present tests which are powerful especially against the alternative that one of the objects has larger probability than the other ones of being ranked high (or low), whilst the other (k-1) objects are ranked in a random order. We denote the sums of the m ranks of each object by

$$(7.1)\qquad \underline{s}_1,\ldots,\underline{s}_k\ ,\quad (m\leqq \underline{s}_i\leqq km).$$

Obviously we have

$$(7.2)\qquad \sum \underline{s}_i = \tfrac{1}{2}mk(k+1).$$

In section 8 the following theorem will be proved.

<u>Theorem 7.1. For each pair $\underline{s}_i,\underline{s}_j$ of the variables (7.1) and for every pair of integers s_i,s_j the following inequality holds under H_o</u>

$$(7.3)\qquad P\left[\underline{s}_i\leqq s_i \text{ and } \underline{s}_j\leqq s_j\right]\leqq P\left[\underline{s}_i\leqq s_i\right].P\left[\underline{s}_j\leqq s_j\right].$$

So we can apply our approximation method for obtaining slippage tests for the variables $\underline{s}_1,\ldots,\underline{s}_k$. Because the marginal distributions of the $\underline{s}_i$ are all equal under H_o, the test statistic for the test against slippage to the right is max $\underline{s}_i$ and for testing against slippage to the left min $\underline{s}_i$. The critical values are determined by the smallest integer S_ε satisfying

(7.4) $$P\left[\underline{s}_i \geqq S_\varepsilon\right] \leqq \varepsilon/k$$

and the largest integer s_ε satisfying

(7.5) $$P\left[\underline{s}_i \leqq s_\varepsilon\right] \leqq \varepsilon/k,$$

respectively.

The distribution of $\underline{s}_i$ is easily seen to be symmetric with respect to the mean value $\frac{1}{2}m(k+1)$, so we have

(7.6) $$s_\varepsilon = m(k+1) - S_\varepsilon .$$

In section 8 it will be shown that the distribution of $\underline{s}_i$, under H_o, reads

(7.7) $$P\left[\underline{s}_i = n\right] = \sum_{x=0}^{\infty} \Big|_{n-kx-m} \binom{m}{x}\binom{n-kx-1}{m-1}(-1)^x k^{-m}, \quad (i=1,\ldots,k; m \leqq n \leqq km)$$ 1)

where $|_y$ is defined by

(7.8) $$\begin{cases} |_y = 0 \text{ if } y \leqq 0, \\ |_y = 1 \text{ if } y > 0. \end{cases}$$

The tables of critical values s_ε, presented in section 11, are based on this formula.

8. Proofs of the results of section 7

First we shall prove theorem 7.1. We suppose that both s_i and s_j are lying between m and km, because otherwise (7.3) obviously holds with the equality sign. For m = 1 we have

(8.1) $$\begin{cases} P\left[\underline{s}_i \leqq s_i \text{ and } \underline{s}_j \leqq s_j \mid m=1\right] = \dfrac{s_i s_j - \min(s_i, s_j)}{k(k-1)}, \\ P\left[\underline{s}_i \leqq s_i \mid m=1\right] = \dfrac{s_i}{k}, \\ P\left[\underline{s}_j \leqq s_j \mid m=1\right] = \dfrac{s_j}{k}, \end{cases}$$

so in that case (7.3) is true. Now let us suppose that (7.3) is true for m observers, then we have

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1) We owe this formula to Mr A. BENARD, Statistical Department of the Mathematical Centre.

$$(8.2)\quad P\left[\underline{s}_i \leqq s_i \text{ and } \underline{s}_j \leqq s_j \mid m+1\right] =$$

$$= \sum_{a \neq b} P\left[\underline{s}_i \leqq s_i - a \text{ and } \underline{s}_j \leqq s_j - b \mid m\right] . P\left[\text{the } i^{th} \text{ object has rank } a \text{ and the } j^{th} \text{ object rank } b \text{ in the } (m+1)^{st} \text{ ranking}\right] =$$

$$= \sum_{a \neq b} P\left[\underline{s}_i \leqq s_i - a \text{ and } \underline{s}_j \leqq s_j - b \mid m\right] . \frac{1}{k(k-1)} \leqq$$

$$\leqq \sum_{a \neq b} P\left[\underline{s}_i \leqq s_i - a \mid m\right] . P\left[\underline{s}_j \leqq s_j - b \mid m\right] . \frac{1}{k(k-1)} =$$

$$= \sum_{a=1}^{k} P\left[\underline{s}_i \leqq s_i - a \mid m\right] . \frac{1}{k} . \sum_{b=1}^{k} P\left[\underline{s}_j \leqq s_j - b \mid m\right] . \frac{1}{k} +$$

$$+ \frac{1}{k^2(k-1)} \sum_{a=1}^{k} P\left[\underline{s}_i \leqq s_i - a \mid m\right] \sum_{b=1}^{k} P\left[\underline{s}_j \leqq s_j - b \mid m\right] +$$

$$- \frac{1}{k(k-1)} \sum_{a=1}^{k} P\left[\underline{s}_i \leqq s_i - a \mid m\right] . P\left[\underline{s}_j \leqq s_j - a \mid m\right] =$$

$$= P\left[\underline{s}_i \leqq s_i \mid m+1\right] . P\left[\underline{s}_j \leqq s_j \mid m+1\right] +$$

$$- \frac{1}{k(k-1)} \sum_{a=1}^{k} \left\{ P\left[\underline{s}_i \leqq s_i - a \mid m\right] - \frac{\sum_{b=1}^{k} P\left[\underline{s}_i \leqq s_i - b \mid m\right]}{k} \right\} .$$

$$. \left\{ P\left[\underline{s}_j \leqq s_j - a \mid m\right] - \frac{\sum_{b=1}^{k} P\left[\underline{s}_j \leqq s_j - b \mid m\right]}{k} \right\} \leqq$$

$$\leqq P\left[\underline{s}_i \leqq s_i \mid m+1\right] . P\left[\underline{s}_j \leqq s_j \mid m+1\right] .$$

So theorem 7.1 is proved by induction.

Formula 7.7 can be proved in the following way:

$k^m P\left[\underline{s}_i = n \mid m\right]$ = the number of partitions of n into m positive integers, no one being larger than k (different permutations of the same integers are counted as different partitions).

Thus

$$k^m P\left[\underline{s}_i = n \mid m\right] = \text{coëfficient of } z^n \text{ in } (z + \ldots + z^k)^m =$$

$$= \text{coëfficient of } z^{n-m} \text{ in } \left(\frac{1-z^k}{1-z}\right)^m = \text{coëfficient of } z^{n-m} \text{ in}$$

$$\sum_{x=0}^{\infty} \binom{m}{x} (-)^x z^{kx} \sum_{r=0}^{\infty} \binom{m+r-1}{r} z^r = \sum_{x=0}^{\infty} {}_{\,n-kx-m}\binom{m}{x} \binom{n-kx-1}{m-1} (-)^x$$

which proves (7.7).

9. A distribution free k-sample slippage test

We consider the independent variates

$$(9.1) \qquad \underline{u}_1, \ldots, \underline{u}_k \,,$$

which have, under H_o, the same continuous distribution function. From the i^{th} population we have t_i independent observations $\underline{u}_{ij}$ $(j=1,\ldots,t_i)$. We want to test H_o against the alternatives

$$(9.2) \qquad H_1 \begin{cases} P\left[\underline{u}_i > \underline{u}_j\right] > \frac{1}{2} \quad (j \neq i), \\ \underline{u}_j \ (j=1,\ldots,i-1,i+1,\ldots,k) \text{ follow the same distribution,} \end{cases}$$

for one unknown value of i and

$$(9.3) \qquad H_2 \begin{cases} P\left[\underline{u}_i > \underline{u}_j\right] < \frac{1}{2} \quad (j \neq i), \\ \underline{u}_j \ (j=1,\ldots,i-1,i+1,\ldots,k) \text{ follow the same distribution.} \end{cases}$$

Now the following test procedure is proposed. If all observations $\underline{u}_{ij}$ $(i=1,\ldots,k;\ j=1,\ldots,t_i)$ are ranked, we denote by $\underline{T}_i$ the sum of the ranks of the observations $\underline{u}_{ij}$ $(j=1,\ldots,t_i)$. As $\underline{T}_i$ is a linear function of WILCOXON's test statistic applied to the i^{th} sample and the other k-1 samples together, its distribution function under H_o is known (cf. H.B. MANN and D.R. WHITNEY (1947)). So for each set of values $T_1,\ldots,T_k$ we can, under H_o, compute

$$(9.4) \qquad q_i = P\left[\underline{T}_i \geqq T_i\right].$$

Now, when testing H_o against H_1, H_o is rejected when $\min q_i \leqq \varepsilon/k$. A similar procedure is followed for slippage to the left. In the next section we shall prove the inequality

$$(9.5) \qquad P\left[\underline{T}_i \geqq T_i \text{ and } \underline{T}_j \geqq T_j\right] \leqq P\left[\underline{T}_i \geqq T_i\right] \cdot P\left[\underline{T}_j \geqq T_j\right],$$

so the limits between which the level of significance may vary are known also in this case.

Let now for every fixed i $H_{1,i}$ be the hypothesis

$$\begin{cases} P\left[\underline{u}_i > \underline{u}_j\right] > \frac{1}{2} \quad (j \neq i), \\ \underline{u}_j (j=1,\ldots,i-1,i+1,\ldots,k), \text{ follow the same distribution.} \end{cases}$$

Put

$$P\left[T_i \mid H_o\right] \stackrel{def}{=} P\left[\underline{T}_i \geqq T_i \mid H_o\right].$$

This probability still depends on $t_1,\ldots,t_k$.

In the same way as in sections 3 and 5 we consider the decision procedure δ:
"Decide that H_o is true if

$$P\left[T_j \mid H_o\right] > \frac{\varepsilon}{k} \quad \text{for } j = 1,\dots,k.$$

Decide that $H_{1,j}$ is true, if j is the smallest integer such that

$$P\left[T_j \mid H_o\right] \leqq \frac{\varepsilon}{k} \text{ and } P\left[T_l \mid H_o\right] \geqq P\left[T_j \mid H_o\right] \quad , \quad l \neq j \;.$$

We prove in the next section

<u>Theorem 9.1. If $H_{1,i}$ is true, the probability of a correct decision with the procedure δ tends to 1 if $t_1 \to \infty, \dots, t_k \to \infty$ such that</u>

$$\liminf \frac{t_i}{\Sigma t_l} > 0 \qquad (i = 1,\dots,k) \;.$$

Another test for the k-sample slippage problem was proposed by F. MOSTELLER (1948) (cf. also F. MOSTELLER and J.W. TUKEY (1950)) who uses as test statistic the number of observations of the sample with the largest observation which exceed all observations of all other samples. A comparison of the power of both tests with respect to some alternatives of practical interest seems desirable.

10. <u>Proof of the inequality (9.5) and of theorem 9.1</u>

For definiteness we take in (9,5) $i = 1$, $j = 2$. We also take $k = 3$. This is no restriction on the generality as pooling of the 3^{rd}, 4^{th}, ... and k^{th} sample does not affect $P\left[T_1 \mid H_o\right]$, $P\left[T_2 \mid H_o\right]$ or $P\left[T_1, T_2 \mid H_o\right] \overset{def}{=}$ $\overset{def}{=} P\left[\underline{T}_1 \geqq T_1 \text{ and } \underline{T}_2 \geqq T_2 \mid H_o\right]$.

Put now

$$t \overset{def}{=} t_1+t_2+t_3 \tag{10.1}$$

and define

$P_{n_1,n_2,n_3}\left[T_i\right] \overset{def}{=} P\left[T_i \mid H_o\right]$ if $t_1=n_1$, $t_2=n_2$, $t_3=n_3$.

$P_{n_1,n_2,n_3}\left[T_i, 1\right] \overset{def}{=} P\left[\underline{T}_i \geqq T_i\right.$ and the largest element belongs to sample number 1 $\left.\mid H_o\right]$ if $t_1=n_1$, $t_2=n_2$, $t_3=n_3$.

$P_{n_1,n_2,n_3}\left[T_i \mid 1\right] \overset{def}{=}$ the conditional probability of $\underline{T}_i \geqq T_i$ under H_o, given that the largest element belongs to sample number 1 if $t_1=n_1$, $t_2=n_2$, $t_3=n_3$.

In the same way we define $P_{n_1,n_2,n_3}\left[T_i, T_j\right]$, $P_{n_1,n_2,n_3}\left[T_i, T_j, 1\right]$ and $P\left[T_i, T_j \mid 1\right]$ for the events $\left\{\underline{T}_i \geqq T_i \text{ and } \underline{T}_j \geqq T_j\right\}$.

We shall prove (9.5) by induction with respect to $n_1+n_2+n_3$. So we have to prove

$$(10.2) \qquad P_{n_1,n_2,n_3}\left[T_1,T_2\right] \leqq P_{n_1,n_2,n_3}\left[T_1\right].P_{n_1,n_2,n_3}\left[T_2\right].$$

Clearly (10.2) holds for $n_1+n_2+n_3 = 2$ (we take $\underline{T}_i = 0$ with probability 1 when $t_i = 0$). Now suppose (10.2) holds if $n_1+n_2+n_3 \leqq t-1$. We have

$$(10.3) \qquad P_{t_1,t_2,t_3}\left[T_1,T_2\right] = \sum_{i=1}^{3} \frac{t_i}{t} P_{t_1,t_2,t_3}\left[T_1,T_2 \mid i\right]$$

For the first term of the sum in the right hand member we get

$$(10.4) \quad P_{t_1,t_2,t_3}\left[T_1,T_2 \mid 1\right] = P_{t_1-1,t_2,t_3}\left[T_1-n,T_2\right] \leqq \text{(according to our assumption)}$$

$$\leqq P_{t_1-1,t_2,t_3}\left[T_1-n\right] P_{t_1-1,t_2,t_3}\left[T_2\right] = P_{t_1,t_2,t_3}\left[T_1 \mid 1\right] P_{t_1,t_2,t_3}\left[T_2 \mid 1\right].$$

In the same way, it can be derived that

$$(10.5) \qquad P_{t_1,t_2,t_3}\left[T_1,T_2 \mid 2\right] \leqq P_{t_1,t_2,t_3}\left[T_1 \mid 2\right].P_{t_1,t_2,t_3}\left[T_2 \mid 2\right].$$

Further

$$(10.6) \qquad P_{t_1,t_2,t_3}\left[T_1,T_2 \mid 3\right] = P_{t_1,t_2,t_3-1}\left[T_1,T_2\right] \leqq$$

$$\leqq P_{t_1,t_2,t_3-1}\left[T_1\right] . P_{t_1,t_2,t_3-1}\left[T_2\right] =$$

$$= P_{t_1,t_2,t_3}\left[T_1 \mid 3\right] P_{t_1,t_2,t_3}\left[T_2 \mid 3\right].$$

So, combining (10.3), (10.4), (10.5) and (10.6) we find, dropping the subscripts

$$(10.7) \qquad P\left[T_1,T_2\right] \leqq \sum_{i=1}^{3} \frac{t_i}{t} P\left[T_1 \mid i\right].P\left[T_2 \mid i\right] = \sum_{i=1}^{3} P\left[T_1 \mid i\right].P\left[T_2, i\right].$$

We have

$$(10.8) \qquad P\left[T_1 \mid 2\right] = P\left[T_2 \mid 3\right] = P\left[T_1 \mid 2 \text{ or } 3\right]$$

and similarly with 1 and 2 interchanged, and

$$(10.9) \qquad P\left[T_1\right] P\left[T_2\right] = \left\{\frac{t_1}{t} P\left[T_1 \mid 1\right] + \frac{t_2+t_3}{t} P\left[T_1 \mid 2 \text{ or } 3\right]\right\} .$$

$$. \left\{P\left[T_2,1\right] + P\left[T_2,2 \text{ or } 3\right]\right\}.$$

From (10.7) and (10.9) we see that it is sufficient to prove

$$(10.10)\quad \sum_{i=1}^{3} P\left[T_1 \mid i\right].P\left[T_2, i\right] = P\left[T_1 \mid 1\right] P\left[T_2, 1\right] + P\left[T_1 \mid 2\right].P\left[T_2, \text{ 2 or 3}\right] \leqq$$

$$\leqq \left\{\frac{t_1}{t} P\left[T_1 \mid 1\right] + \frac{t_2+t_3}{t} P\left[T_1 \mid \text{2 or 3}\right]\right\} \left\{P\left[T_2, 1\right] + P\left[T_2, \text{ 2 or 3}\right]\right\}$$

or its equivalent

$$(10.11)\quad \left\{P\left[T_1 \mid 1\right] - P\left[T_1 \mid 2\right]\right\} \left\{\frac{t_2+t_3}{t} P\left[T_2, 1\right] - \frac{t_1}{t} P\left[T_2, \text{ 2 or 3}\right]\right\} \lesseqqgtr 0 .$$

But the inequality

$$(10.12)\quad P\left[T_1 \mid 1\right] \geqq P\left[T_1 \mid 2\right]$$

holds as can be seen in the following way
(10.12) is equivalent to

$$(10.13)\quad t_1 P\left[T_1, 2\right] \leqq t_2 P\left[T_1, 1\right] .$$

Consider now a ranking which gives T_1 and 2 (i.e. the largest element belongs to the 2[nd] sample and $\underline{T}_1 \geqq T_1$) and interchange the last element with every element of the first sample. This gives t_1 different rankings with T_1 and 1. In this way we get each ranking with T_1 and 1 at most t_2 times, because in a ranking with T_1 and 1 the last element can be interchanged with at most t_2 different elements of the second sample. This proves (10.13) and thus (10.12). Interchanging 1 and 2 in (10.12) we find

$$(10.14)\quad P\left[T_2 \mid 2\right] \geqq P\left[T_2 \mid 1\right].$$

(10.11) and thus (10.2) is an immediate consequence of (10.12) and (10.14). This completes the proof of (9.5).

We now turn to the proof of theorem 9.1. Let $H_{1,1}$ be true. If $t_i \to \infty$ $(i = 1, \ldots, k)$ such that

$$\liminf \frac{t_1}{\sum_{i=1}^{k} t_i} > 0 \text{ and } \liminf \frac{\sum_{i=1}^{k} t_i - t_1}{\sum t_i} > 0,$$

we know that Wilcoxon's test comparing sample 1 with the other samples pooled is consistent. This means

$$(10.15)\quad \lim_{t_i \to \infty} P\left[\, P\left[\underline{T}_1\right] \leqq \eta \mid H_{1,1}\right] = 1$$

for every η $(0 \leqq \eta \leqq 1)$

From (10.7) and (10.9) we see that it is sufficient to prove

$$(10.10) \quad \sum_{i=1}^{3} P\left[T_1 \mid i\right].P\left[T_2, i\right] = P\left[T_1 \mid 1\right] P\left[T_2, 1\right] + P\left[T_1 \mid 2\right].P\left[T_2,\ 2 \text{ or } 3\right] \leqq$$

$$\leqq \left\{\frac{t_1}{t} P\left[T_1 \mid 1\right] + \frac{t_2+t_3}{t} P\left[T_1 \mid 2 \text{ or } 3\right]\right\} \left\{P\left[T_2, 1\right] + P\left[T_2,\ 2 \text{ or } 3\right]\right\}$$

or its equivalent

$$(10.11) \quad \left\{P\left[T_1 \mid 1\right] - P\left[T_1 \mid 2\right]\right\} \left\{\frac{t_2+t_3}{t} P\left[T_2, 1\right] - \frac{t_1}{t} P\left[T_2,\ 2 \text{ or } 3\right]\right\} \leqq 0 .$$

But the inequality

$$(10.12) \qquad P\left[T_1 \mid 1\right] \geqq P\left[T_1 \mid 2\right]$$

holds as can be seen in the following way
(10.12) is equivalent to

$$(10.13) \qquad t_1 P\left[T_1, 2\right] \leqq t_2 P\left[T_1, 1\right] .$$

Consider now a ranking which gives T_1 and 2 (i.e. the largest element belongs to the 2^{nd} sample and $\underline{T}_1 \geqq T_1$) and interchange the last element with every element of the first sample. This gives t_1 different rankings with T_1 and 1. In this way we get each ranking with T_1 and 1 at most t_2 times, because in a ranking with T_1 and 1 the last element can be interchanged with at most t_2 different elements of the second sample. This proves (10.13) and thus (10.12). Interchanging 1 and 2 in (10.12) we find

$$(10.14) \qquad P\left[T_2 \mid 2\right] \geqq P\left[T_2 \mid 1\right].$$

(10.11) and thus (10.2) is an immediate consequence of (10.12) and (10.14). This completes the proof of (9.5).

We now turn to the proof of theorem 9.1. Let $H_{1,1}$ be true. If $t_i \to \infty (i = 1, \ldots, k)$ such that

$$\liminf \frac{t_1}{\sum_{i=1}^{k} t_i} > 0 \text{ and } \liminf \frac{\sum_{i=1}^{k} t_i - t_1}{\sum t_i} > 0,$$

we know that Wilcoxon's test comparing sample 1 with the other samples pooled is consistent. This means

$$(10.15) \qquad \lim_{t_i \to \infty} P\left[P\left[\underline{T}_1 \right] \leqq \eta \mid H_{1,1} \right] = 1$$

for every η $(0 \leqq \eta \leqq 1)$

or the exceedance probability found in the first sample converges to 0 in probability (cf. D. VAN DANTZIG (1951)).

In a similar way as in D. VAN DANTZIG (1951) we find, if

$$p \overset{def}{=} P(\underline{u}_1 > u_j | H_{1,1}) > \tfrac{1}{2}$$

(10.16) $E(\underline{T}_j | H_o) = \tfrac{1}{2}t_j(\sum t_i - t_j) + \tfrac{1}{2}t_j(t_j+1)$

and

(10.17) $E(\underline{T}_j / H_{1,1}) = \tfrac{1}{2}t_j(\sum t_i - t_j - t_1) + (1-p)t_j t_1 + \tfrac{1}{2}t_j(t_j+1) < E(\underline{T}_j | H_o)$

Further

(10.18) $$\sigma^2(\underline{T}_j | H_{1,1}) \leqq 3\sigma^2(\underline{T}_j / H_o).$$

From (10.15) we have

(10.19) $$\lim_{t_i \to \infty} P\left[P[\underline{T}_j] \leqq P[\underline{T}_1] \; \middle| \; H_{1,1}\right] \leqq \lim_{t_i \to \infty} P\left[P[\underline{T}_j] \leqq \eta | H_{1,1}\right]$$

for every η $(0 \leqq \eta \leqq 1)$.

As the limit distribution under H_o of $\dfrac{\underline{T}_j - E(\underline{T}_j \mid H_o)}{\sigma(\underline{T}_j | H_o)}$ is normal with mean 0 and unit variance (10.19) leads to

(10.20) $$\lim_{t_i \to \infty} P\left[P[\underline{T}_j] \leqq \eta | H_{1,1}\right] = \lim_{t_i \to \infty} P\left[\frac{\underline{T}_j - E(\underline{T}_j | H_o)}{\sigma(\underline{T}_j | H_o)} \geqq \xi_\eta \middle| H_{11}\right] \leqq$$

$$\leqq \lim_{t_i \to \infty} P\left[\frac{\underline{T}_j - E(\underline{T}_j | H_{1,1})}{\sigma(\underline{T}_j | H_{1,1})} \geqq \sqrt{3}\,\xi_\eta \;\middle|\; H_{1,1}\right] \leqq \frac{1}{3\xi_\eta^2}$$

where ξ_η is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_\eta}^{\infty} e^{-\frac{x^2}{2}} dx = \eta.$$

(10.20) is valid for every η $(0 \leqq \eta \leqq 1)$ and as $\xi_\eta \to \infty$ $(\eta \to 0)$ (10.19) combined with (10.20) gives

(10.21) $$\lim_{t_i \to \infty} P\left[P[\underline{T}_j] \leqq P[\underline{T}_1] \;\middle|\; H_{1,1}\right] = 0.$$

If $H_{1,1}$ is true the probability of correct decision is

(10.22) $$P\left[P[\underline{T}_1] \leqq \tfrac{\varepsilon}{k} \text{ and } P[\underline{T}_1] \geqq P[\underline{T}_j] \text{ for } j \neq 1 \;\middle|\; H_{1,1}\right] \geqq$$

$$\geqq P\left[P[\underline{T}_1] \leqq \tfrac{\varepsilon}{k} \;\middle|\; H_{1,1}\right] - \sum_{j=2}^{k} P\left[P[\underline{T}_j] > P[\underline{T}_1] \;\middle|\; H_{1,1}\right].$$

(10.15) and (10.21) show that the probability of a correct decision converges to 1, which proves theorem 9.1.

11. Tables of critical values for the Poisson distribution and for the method of m rankings

Table 11.1 gives critical values for the test for Poisson variates against slippage to the right if H_0 is: $p_1 = p_2 = \dots = p_k$. The critical values for max z_i as test statistic are given for the values of ε 0,05 (the upper numbers) and 0,01 (the lower numbers). Owing to the discontinuous character of the binomial distribution the true level of significance will generally be less, and very often considerably less, than ε. Therefore approximated levels of significance (i.e. ε' cf. p. 37) are shown also. The exact values satisfy inequality (2.13). The table was constructed with the help of a table of the binomial distribution. This can also be done for critical values for the test against slippage to the left. Table 11.2 gives critical values for specified ε for the method of m rankings, when testing against slippage to the left with min s_i as test statistic. If this critical value is equal to 1, the critical value r at the same level of significance for the test against slippage to the right is given by $r = m(k+1) - 1$.

As in table 11.1 the approximated true levels of significance (ε') are also given.

Table 11.1

Critical values for the slippage test to the right in the Poisson-case with $H_o: \mu_1 = \mu_2 = \ldots = \mu_k$. Test statistic: max z_i. Approximate significance level 0.05 (upper values) and 0.01 (lower values). The approximated true level of significance is written behind the critical value. Number of observations k, sum of the observations n.

n \ k	2		3		4		5		6		7		8		9		10	
2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	3	0.040	3	0.028	3	0.020	3	0.016	3	0.012	3	0.010
	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3	0.010
4	-	-	4	0.037	4	0.016	4	0.008	4	0.005	4	0.003	4	0.002	3	0.045	3	0.037
	-	-	-	-	-	-	4	0.008	4	0.005	4	0.003	4	0.002	4	0.001	4	0.001
5	-	-	5	0.012	5	0.004	4	0.034	4	0.020	4	0.013	4	0.009	4	0.006	4	0.005
	-	-	-	-	5	0.004	5	0.002	5	0.001	5	0.000	4	0.009	4	0.006	4	0.005
6	6	0.031	6	0.004	5	0.019	5	0.008	5	0.004	4	0.035	4	0.024	4	0.017	4	0.013
	-	-	6	0.004	6	0.001	5	0.008	5	0.004	5	0.002	5	0.001	5	0.001	5	0.001
7	7	0.016	6	0.021	6	0.005	5	0.023	5	0.012	5	0.007	5	0.004	4	0.037	4	0.027
	-	-	7	0.001	6	0.005	6	0.002	6	0.001	5	0.007	5	0.004	5	0.003	5	0.002
8	8	0.008	7	0.008	6	0.017	6	0.006	5	0.028	5	0.016	5	0.010	5	0.006	5	0.004
	8	0.008	7	0.008	7	0.002	6	0.006	6	0.003	6	0.001	5	0.010	5	0.006	5	0.004
9	8	0.039	7	0.025	6	0.040	6	0.015	6	0.007	5	0.032	5	0.020	5	0.013	5	0.009
	9	0.004	8	0.003	7	0.005	7	0.002	6	0.007	6	0.003	6	0.002	6	0.001	5	0.009
10	9	0.021	8	0.010	7	0.014	6	0.032	6	0.015	6	0.008	5	0.036	5	0.024	5	0.016
	10	0.002	9	0.001	8	0.002	7	0.004	7	0.002	6	0.008	6	0.004	6	0.002	6	0.001
11	10	0.012	8	0.027	7	0.030	7	0.010	6	0.028	6	0.015	6	0.008	5	0.040	5	0.028
	11	0.001	9	0.004	8	0.005	7	0.010	7	0.004	7	0.002	6	0.008	6	0.005	6	0.003
12	10	0.039	9	0.012	8	0.011	7	0.020	6	0.048	6	0.026	6	0.015	6	0.009	5	0.043
	11	0.006	10	0.002	9	0.002	8	0.003	7	0.008	7	0.004	7	0.002	6	0.009	6	0.005
13	11	0.022	9	0.027	8	0.023	7	0.035	7	0.015	6	0.042	6	0.024	6	0.015	6	0.009
	12	0.003	10	0.005	9	0.004	8	0.006	8	0.002	7	0.007	7	0.003	7	0.002	6	0.009
14	12	0.013	10	0.012	8	0.041	8	0.012	7	0.025	7	0.012	6	0.038	6	0.023	6	0.015
	13	0.002	11	0.002	9	0.009	9	0.002	8	0.004	8	0.002	7	0.006	7	0.003	7	0.002
15	12	0.035	10	0.026	9	0.017	8	0.021	7	0.040	7	0.019	7	0.010	6	0.035	6	0.022
	13	0.007	11	0.005	10	0.003	9	0.004	8	0.008	8	0.003	8	0.001	7	0.005	7	0.003
16	13	0.021	10	0.048	9	0.030	8	0.035	8	0.013	7	0.030	7	0.016	7	0.009	6	0.033
	14	0.004	12	0.002	10	0.007	9	0.007	9	0.002	8	0.005	8	0.002	7	0.009	7	0.005
17	13	0.049	11	0.024	9	0.050	9	0.013	8	0.021	7	0.045	7	0.024	7	0.013	6	0.047
	15	0.002	12	0.006	11	0.002	10	0.002	9	0.004	8	0.009	8	0.004	8	0.002	7	0.008
18	14	0.031	11	0.044	10	0.022	9	0.021	8	0.032	8	0.014	7	0.035	7	0.020	7	0.012
	15	0.008	13	0.003	11	0.005	10	0.005	9	0.007	9	0.003	8	0.007	8	0.003	8	0.002
19	15	0.019	12	0.022	10	0.036	9	0.033	8	0.048	8	0.021	7	0.050	7	0.028	7	0.017
	16	0.004	13	0.006	11	0.009	10	0.008	10	0.002	9	0.004	9	0.002	8	0.005	8	0.003
20	15	0.041	12	0.039	11	0.016	9	0.050	9	0.017	8	0.031	8	0.015	7	0.040	7	0.024
	17	0.003	14	0.003	12	0.004	11	0.003	10	0.004	9	0.007	9	0.003	8	0.008	8	0.004
21	16	0.027	13	0.021	11	0.026	10	0.020	9	0.026	8	0.044	8	0.022	8	0.011	7	0.033
	17	0.007	14	0.006	12	0.007	11	0.005	10	0.006	10	0.002	9	0.004	9	0.002	8	0.006
22	17	0.017	13	0.035	11	0.040	10	0.031	9	0.037	9	0.015	8	0.031	8	0.016	7	0.044
	18	0.004	15	0.003	13	0.003	11	0.008	10	0.009	10	0.003	9	0.007	9	0.003	8	0.009
23	17	0.035	14	0.019	12	0.019	10	0.045	10	0.014	9	0.022	8	0.042	8	0.022	8	0.012
	19	0.003	15	0.005	13	0.005	12	0.003	11	0.003	10	0.005	9	0.010	9	0.004	9	0.002
24	18	0.023	14	0.031	12	0.029	11	0.019	10	0.020	9	0.030	9	0.014	8	0.030	8	0.017
	19	0.007	15	0.010	13	0.008	12	0.005	11	0.005	10	0.007	10	0.003	9	0.006	9	0.003
25	18	0.043	14	0.049	12	0.043	11	0.028	10	0.029	9	0.041	9	0.019	8	0.040	8	0.023
	20	0.004	16	0.005	14	0.004	12	0.008	11	0.008	11	0.002	10	0.004	9	0.009	9	0.005

Table 11.2

Critical values s_c of the test statistic $\min s_i$ for the slippage test to the left for the method of m rankings. Level of significance ε, number of rankings m, number of ranked objects k. The approximated true levels of significance are written behind the corresponding critical values.

k	ε \ m	3	4	5	6	7	8	9
2	0.05	- -	- -	- -	6 0.031	7 0.016	8 0.008	10 0.039
	0.025	- -	- -	- -	- -	7 0.016	8 0.008	9 0.004
	0.01	- -	- -	- -	- -	- -	8 0.008	9 0.004
3	0.05	- -	4 0.037	5 0.012	7 0.029	9 0.049	10 0.021	12 0.032
	0.025	- -	- -	5 0.012	6 0.004	8 0.011	10 0.021	11 0.008
	0.01	- -	- -	- -	6 0.004	7 0.001	9 0.004	11 0.008
4	0.05	- -	4 0.016	6 0.023	8 0.027	10 0.029	12 0.030	14 0.029
	0.025	- -	4 0.016	6 0.023	7 0.007	9 0.009	11 0.010	13 0.011
	0.01	- -	- -	5 0.004	7 0.007	9 0.009	10 0.003	12 0.003
5	0.05	3 0.040	5 0.040	7 0.034	9 0.027	11 0.021	14 0.038	16 0.028
	0.025	- -	4 0.008	6 0.010	8 0.009	11 0.021	13 0.016	15 0.013
	0.01	- -	4 0.008	6 0.010	8 0.009	10 0.008	12 0.006	14 0.005
6	0.05	3 0.028	5 0.023	8 0.043	10 0.027	13 0.037	16 0.045	18 0.028
	0.025	- -	5 0.023	7 0.016	9 0.011	12 0.017	15 0.023	17 0.014
	0.01	- -	4 0.005	6 0.005	8 0.004	11 0.007	13 0.005	16 0.007
7	0.05	3 0.020	6 0.044	8 0.023	11 0.027	14 0.029	17 0.029	21 0.048
	0.025	3 0.020	5 0.014	8 0.023	10 0.012	13 0.015	16 0.016	19 0.016
	0.01	- -	4 0.003	7 0.009	9 0.005	12 0.007	15 0.008	18 0.008
8	0.05	3 0.016	6 0.029	9 0.031	12 0.028	16 0.043	19 0.035	23 0.046
	0.025	3 0.016	5 0.010	8 0.014	11 0.014	15 0.025	18 0.021	21 0.017
	0.01	- -	5 0.010	7 0.005	10 0.006	13 0.007	16 0.006	20 0.010
9	0.05	4 0.049	7 0.048	10 0.038	13 0.029	17 0.036	21 0.042	25 0.045
	0.025	3 0.012	6 0.021	9 0.019	12 0.016	16 0.022	19 0.016	23 0.019
	0.01	- -	5 0.007	8 0.009	11 0.008	14 0.006	18 0.009	21 0.007
10	0.05	4 0.040	7 0.035	11 0.046	14 0.030	18 0.032	23 0.048	27 0.045
	0.025	3 0.010	6 0.015	9 0.013	13 0.017	17 0.019	21 0.020	25 0.020
	0.01	3 0.010	5 0.005	8 0.006	12 0.009	15 0.006	19 0.008	23 0.008

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